Contents 27

Multiple Integration

Learning outcomes

In this Workbook you will learn to integrate a function of two variables over various rectangular and non-rectangular areas. You will learn how to do this for various other coordinate systems. You will learn to integrate a function of three variables over a volume.

Introduction to Surface Integrals

27.1

Introduction

Often in Engineering it is necessary to find the sum of a quantity over an area or surface. This can be achieved by means of a surface integral also known as a double integral i.e. a function is integrated twice, once with respect to one variable and subsequently with respect to another variable. This Section looks at the concept of the double integral and how to evaluate a double integral over a rectangular area.

Prerequisites

Before starting this Section you should ...

- thoroughly understand the various techniques of integration
- be familiar with the concept of a function of two variables

Learning Outcomes

On completion you should be able to ...

- understand the concept of a surface integral
- integrate a function over a rectangular region

1. An example of a surface integral

An engineer involved with the construction of a dam to hold back the water in a reservoir needs to be able to calculate the total force the water exerts on the dam so that the dam is built with sufficient strength.
In order to calculate this force, two results are required:

(a) The pressure p of the water is proportional to the depth. That is

$$p = kd \tag{1}$$

where k is a constant.

(b) The force on an area subjected to constant pressure is given by

$$\text{force} = \text{pressure} \times \text{area} \tag{2}$$

The diagram shows the face of the dam. The depth of water is h and δA is a small area in the face of the dam with coordinates (x, y).

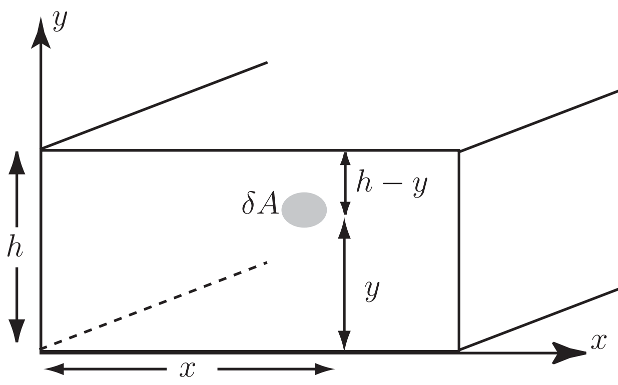


Figure 1

Using (1), the pressure at $\delta A \sim k(h-y)$. Using (2), the force on an area $\delta A \sim k(h-y)\delta A$. Both of these expressions are approximate as y is slightly different at the top of δA to the bottom. Now

$$\begin{aligned}\text{Total force on dam} &= \text{sum of forces on all areas } \delta A \text{ making up the face of the dam}\\ &\approx \sum_{\text{all } \delta A} k(h-y)\delta A\end{aligned}$$

For a better approximation let δA become smaller, and for the exact result find the limit as $\delta A \to 0$. Then

$$\begin{aligned}\text{Total force on the dam} &= \lim_{\delta A \to 0} \sum k(h-y)\delta A\\ &= \int_A k(h-y)\ dA\end{aligned}$$

where $\displaystyle\int_A k(h-y)\ dA$ stands for the **surface integral** of $k(h-y)$ over the area A. Surface integrals are evaluated using **double integrals**. The following Section shows a double integral being developed in the case of the volume under a surface.

2. Single and double integrals

As has been seen in HELM 14.3, the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$ (assuming that the curve lies above the axis for all x in the range $a \le x \le b$). This is illustrated by the figure below.

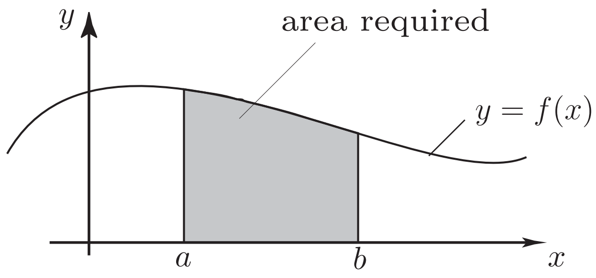


Figure 2

In a similar manner, the volume under a surface (given by a function of two variables $z = f(x,y)$) and above the xy plane can be found by integrating the function $z = f(x,y)$ twice, once with respect to x and once with respect to y.

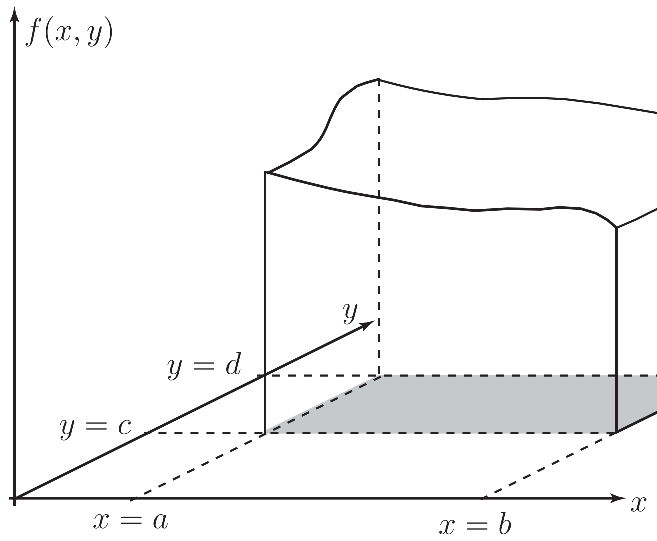


Figure 3

The above figure shows the part of a surface given by $f(x,y)$ which lies above the rectangle $a \le x \le b$, $c \le y \le d$. This rectangle is shaded and the volume above this rectangle but below the surface can be seen.

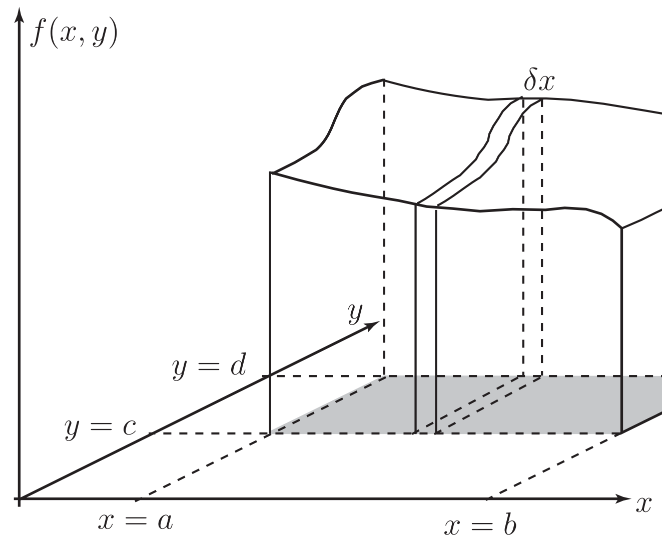


Figure 4

Imagine a vertical slice taken through this volume at right angles to the x-axis (figure above). This slice has thickness δx and lies at position x. Assuming that δx is small enough that the areas of both sides (left and right) of this slice are virtually the same, the area of each face of the slice is given by the integral

$$\int_{y=c}^{y=d} f(x,y)\ dy \qquad \text{(where } x \text{ measures the position of the slice)}$$

and the volume of the slice will be given by

$$\delta x \int_{y=c}^{y=d} f(x,y)\ dy$$

To find the total volume between the surface and the xy plane, this quantity should be summed over all possible such slices, each for a different value of x. Thus

$$V \approx \sum_i \int_{y=c}^{y=d} f(x_i,y)\ dy\ \delta x$$

When δx becomes infinitesmally small, it can be considered to be dx and the summation will change into an integral. Hence

$$V = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \quad dydx$$

Thus the volume is given by integrating the function twice, once with respect to x and once with respect to y.

The procedure shown here considers the volume above a rectangular area and below the surface. The volume beneath the surface over a non-rectangular area can also be found by integrating twice (see Section 27.2).

Key Point 1

Volume Integral

The volume under the surface $z = f(x, y)$ and above a rectangular region in the xy plane (that is the rectangle $a \leq x \leq b, \quad c \leq y \leq d$) is given by the integral:

$$V = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y) \;\; dydx$$

3. 'Inner' and 'Outer' integrals

A typical double integral may be expressed as

$$I = \int_{x=a}^{x=b} \quad \left[\int_{y=c}^{y=d} f(x, y) \;\; dy \right] \; dx$$

where the part in the centre i.e.

$$\int_{y=c}^{y=d} f(x, y) \;\; dy$$

(known as the inner integral) is the integral of a function of x and y with respect to y. As the integration takes place with respect to y, the variable x may be regarded as a fixed quantity (a constant) but for every different value of x, the inner integral will take a different value. Thus, the inner integral will be a function of x e.g. $g(x) = \displaystyle\int_{y=c}^{y=d} f(x, y) \;\; dy$.

This inner integral, being a function of x, once evaluated, can take its place within the outer integral i.e. $I = \displaystyle\int_{x=a}^{x=b} g(x) \;\; dx$ which can then be integrated with respect to x to give the value of the double integral.

The limits on the outer integral will be constants; the limits on the inner integral may be constants (in which case the integration takes place over a rectangular area) or may be functions of the variable used for the outer integral (in this case x). In this latter case, the integration takes place over a non-rectangular area (see Section 27.2). In the Examples quoted in this Section or in the early parts of the next Section, the limits include the name of the relevant variable; this can be omitted once more familiarity has been gained with the concept. It will be assumed that the limits on the inner integral apply to the variable used to integrate the inner integral and the limits on the outer integral apply to the variable used to integrate this outer integral.

4. Integration over rectangular areas

Consider the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} (2x + y) \quad dydx$$

This represents an integral over the rectangle shown below.

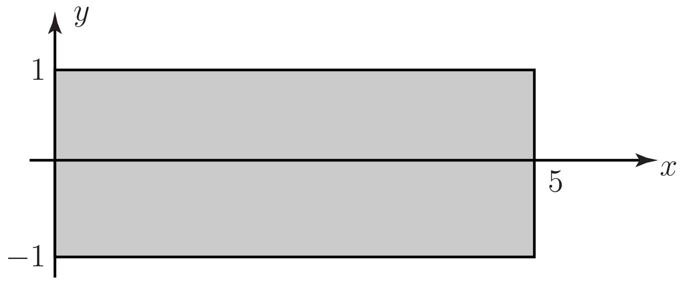


Figure 5

Here, the inner integral is

$$g\,(x) = \int_{-1}^{1} (2x + y)\ dy$$

and the outer integral is

$$I = \int_{x=0}^{5} g\,(x)\ dx$$

Looking in more detail at the inner integral

$$g\,(x) = \int_{-1}^{1} (2x + y)\ dy$$

the function $(2x + y)$ can be integrated with respect to y (keeping x constant) to give $2xy + \frac{1}{2}y^2 + C$ (where C is a constant and can be omitted as the integral is a definite integral) i.e.

$$g\,(x) = \left[2xy + \frac{1}{2}y^2\right]_{-1}^{1} = \left(2x + \frac{1}{2}\right) - \left(-2x + \frac{1}{2}\right) = 2x + \frac{1}{2} + 2x - \frac{1}{2} = 4x.$$

This is a function of x as expected. This inner integral can be placed into the outer integral to get

$$I = \int_{x=0}^{5} 4x \ \ dx$$

which becomes

$$I = \left[\ 2x^2\right]_{0}^{5} = 2 \times 5^2 - 2 \times 0^2 = 2 \times 25 - 0 = 50$$

Hence the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} (2x + y) \quad dydx = 50$$

Key Point 2

Double Integral

When evaluating a double integral, evaluate the inner integral first and substitute the result into the outer integral.

Example 1

Evaluate the double integral $I = \displaystyle\int_{x=-1}^{2}\int_{y=-2}^{3} x^2 y \; dydx$

This integral is evaluated over the area shown below.

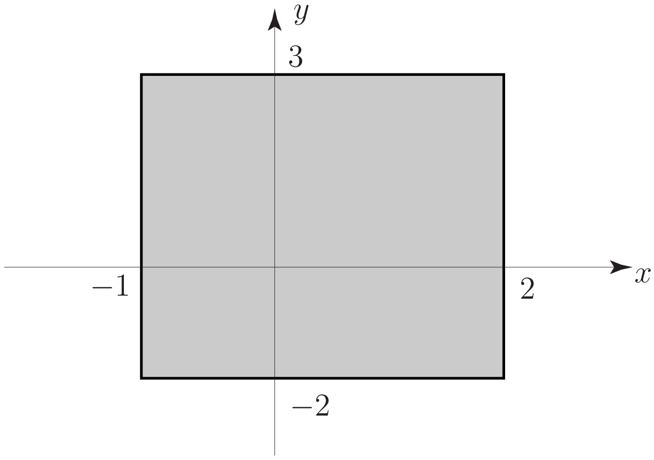


Figure 6

Solution

Here, the inner integral is

$$g\left(x\right) = \int_{y=-2}^{3} x^2 y \; dy = \left[x^2\frac{y^2}{2}\right]_{-2}^{3} = \frac{9}{2}x^2 - \frac{4}{2}x^2 = \frac{5}{2}x^2$$

and hence the outer integral is

$$I = \int_{x=-1}^{2} \frac{5}{2}x^2 \, dx = \left[\frac{5}{2}\frac{1}{3}x^3\right]_{-1}^{2} = \frac{5}{6}\times 8 - \frac{5}{6}\left(-1\right) = \frac{15}{2}$$

Example 2

Use the above approach to evaluate the double integral

$$I = \int_{x=0}^{5} \int_{y=-1}^{1} x^2 \cos \frac{\pi y}{2} \ dydx$$

Note that the limits are the same as in a previous case but that the function itself has changed.

Solution

The inner integral is

$$\int_{y=-1}^{1} x^2 \cos \frac{\pi y}{2} \ dy = \left[\frac{2}{\pi} x^2 \sin \frac{\pi y}{2} \right]_{-1}^{1} = \frac{2}{\pi} x^2 1 - \frac{2}{\pi} x^2 (-1) = \frac{4}{\pi} x^2$$

so the outer integral becomes

$$I = \int_{x=0}^{5} \frac{4}{\pi} x^2 \ dx = \left[\frac{4}{3\pi} x^3 \right]_0^5 = \frac{4}{3\pi} 125 - \frac{4}{3\pi} 0 = \frac{500}{3\pi} \approx 53.1$$

Clearly, variables other than x and y may be used.

Example 3

Evaluate the double integral

$$I = \int_{s=1}^{4} \int_{t=0}^{\pi} s \sin t \ dtds$$

Solution

This integral becomes (dispensing with the step of formally writing the inner integral),

$$\begin{aligned} I &= \int_{s=1}^{4} \left[-s \cos t \right]_0^{\pi} ds = \int_1^4 \left[-s \cos \pi + s \cos 0 \right] ds = \int_1^4 \left[-s(-1) + s(1) \right] ds \\ &= \int_1^4 2s \ ds = \left[s^2 \right]_1^4 = 16 - 1 = 15 \end{aligned}$$

Clearly, evaluating the integrals can involve further tools of integration, e.g. integration by parts or by substitution.

Example 4

Evaluate the double integral

$$I = \int_{-1}^{2}\int_{-2}^{3} \frac{xye^{-x}}{y^2+1} \quad dydx$$

Here, the limits have not formally been linked with a variable name but the limits on the outer integral apply to x and the limits on the inner integral apply to y. As the integrations are more complicated, the inner integral will be evaluated explicitly.

Solution

$$\text{Inner integral } = \int_{-2}^{3} \frac{xye^{-x}}{y^2+1} \; dy$$

which can be evaluated by means of the substitution $U = y^2 + 1$.

If $\quad U = y^2 + 1 \quad$ then $\quad dU = 2y\, dy \quad$ so $\quad y\, dy = \dfrac{1}{2} dU$.

Also if $\quad y = -2 \quad$ then $\quad U = 5 \quad$ and if $\quad y = 3 \quad$ then $\quad U = 10.$

So the inner integral becomes (remembering that x may be treated as a constant)

$$\int_{5}^{10} \frac{1}{2}\frac{xe^{-x}}{U}\; dU = \frac{xe^{-x}}{2}\int_{5}^{10} \frac{dU}{U} = \frac{xe^{-x}}{2}\Big[\ln U\Big]_5^{10} = \frac{xe^{-x}}{2}(\ln 10 - \ln 5) = xe^{-x}\frac{\ln 2}{2}$$

and so the double integral becomes

$$I = \int_{-1}^{2} xe^{-x}\frac{\ln 2}{2}\; dx = \frac{\ln 2}{2}\int_{-1}^{2} xe^{-x}\; dx$$

which can be evaluated by integration by parts.

$$\begin{aligned} I &= \frac{\ln 2}{2}\left[\Big[-xe^{-x}\Big]_{-1}^{2} - \int_{-1}^{2} 1 \times \left(-e^{-x}\right)\; dx\right] = \frac{\ln 2}{2}\left[-2e^{-2} + (-1)\,e^{1} + \int_{-1}^{2} e^{-x}\; dx\right] \\ &= \frac{\ln 2}{2}\left[-2e^{-2} - e^{1} + \Big[-e^{-x}\Big]_{-1}^{2}\right] \\ &= \frac{\ln 2}{2}\left[-2e^{-2} - e^{-1} - e^{-2} + e^{1}\right] = \frac{\ln 2}{2}\left[-3e^{-2}\right] \approx -0.14 \end{aligned}$$

Task

Evaluate the following double integral.

$$I = \int_{-1}^{1} \int_{0}^{2} \left(x^2y + 3y^2\right) \quad dydx$$

Your solution

Answer

The inner integral $= \displaystyle\int_0^2 \left(x^2y + 3y^2\right) dy = \left[\frac{1}{2}x^2y^2 + y^3\right]_0^2 = \frac{1}{2} \times 4x^2 + 8 - (0+0) = 2x^2 + 8$

This can be put in the outer integral to give

$I = \displaystyle\int_{-1}^{1} \left(2x^2 + 8\right) \ dx = \left[\frac{2}{3}x^3 + 8x\right]_{-1}^{1} = \frac{2}{3} + 8 - (-\frac{2}{3} - 8) = \frac{4}{3} + 16 = \frac{52}{3}$

Exercises

Evaluate the following double integrals over rectangular areas.

1. $I = \displaystyle\int_{x=0}^{1} \int_{y=0}^{2} xy \quad dydx$
2. $I = \displaystyle\int_{-2}^{3} \int_{0}^{4} \left(x^2 + y^2\right) \quad dxdy$
3. $I = \displaystyle\int_{0}^{\pi} \int_{-1}^{1} y \sin^2 x \quad dydx$
4. $I = \displaystyle\int_{0}^{2} \int_{-1}^{3} st^3 \ dsdt$
5. $I = \displaystyle\int_{0}^{3} \int_{0}^{1} 5z^2w \left(w^2 - 1\right)^4 \ dwdz$ (Requires integration by substitution.)
6. $I = \displaystyle\int_{0}^{2\pi} \int_{0}^{1} ty \sin t \quad dydt$ (Requires integration by parts.)

Answers

1. 1, 2. 460/3, 3. 0, 4. 16, 5. 9/2, 6. $-\pi$

5. Special cases

If the integrand can be written as

$$f(x, y) = g(x)\, h(y)$$

then the double integral

$$I = \int_a^b \int_c^d g(x)\, h(y) \quad dydx$$

can be written as

$$I = \int_a^b g(x)\ dx \times \int_c^d h(y)\ dy$$

i.e. the **product** of the two individual integrals. For example, the integral

$$I = \int_{x=-1}^{2} \int_{y=-2}^{3} x^2 y \quad dydx$$

which was evaluated earlier can be written as

$$I = \int_{x=-1}^{2} x^2\, dx \times \int_{y=-2}^{3} y\, dy = \left[\frac{x^3}{3}\right]_{-1}^{2} \left[\frac{y^2}{2}\right]_{-2}^{3} = \left[\frac{8}{3} - \frac{(-1)}{3}\right] \left[\frac{9}{2} - \frac{4}{2}\right]$$

$$= 3 \times \frac{5}{2} = \frac{15}{2}$$

the same result as before.

Key Point 3

Double Integral as a Product

The integral

$$\int_a^b \int_c^d g(x)h(y) \quad dydx \quad \text{can be written as} \quad \int_a^b g(x)\, dx \quad \times \quad \int_c^d h(y)\, dy$$

Imagine the integral

$$I = \int_{-1}^{1} \int_0^1 x\mathrm{e}^{-y^2} \quad dydx$$

Approached directly, this would involve evaluating the integral $\displaystyle\int_0^1 x\mathrm{e}^{-y^2}\ dy$ which cannot be done by algebraic means (i.e. it can only be determined numerically).

However, the integral can be re-written as

$$I = \int_{-1}^{1} x \, dx \times \int_{0}^{1} \mathrm{e}^{-y^2} \, dy = \left[\frac{1}{2}x^2 \right]_{-1}^{1} \times \int_{0}^{1} \mathrm{e}^{-y^2} \, dy = 0 \times \int_{0}^{1} \mathrm{e}^{-y^2} \, dy = 0$$

and the result can be found without the need to evaluate the difficult integral.
If the integrand is independent of one of the variables and is simply a function of the other variable, then only one integration need be carried out.

The integral $I_1 = \int_a^b \int_c^d h\,(y) \quad dydx$ may be written as $I_1 = (b-a) \int_c^d h\,(y) \; dy$ and the integral $I_2 = \int_a^b \int_c^d g\,(x) \quad dydx$ may be written as $I_2 = (d-c) \int_a^b g\,(x) \; dx$ i.e. the integral in the variable upon which the integrand depends multiplied by the length of the range of integration for the other variable.

Example 5

Evaluate the double integral

$$I = \int_0^2 \int_{-1}^{2} y^2 \quad dydx$$

Solution

As the integral in y can be multiplied by the range of integration in x, the double integral will equal

$$I = (2-0) \int_{-1}^{2} y^2 \, dy = 2 \left[\frac{y^3}{3} \right]_{-1}^{2} = 2 \left[\frac{2^3}{3} - \frac{(-1)^3}{3} \right] = 6$$

Note that the two integrations can be carried out in either order as long as the limits are associated with the correct variable. For example

$$I = \int_{x=0}^{1} \int_{y=-1}^{2} x^4 y \quad dydx = \int_{x=0}^{1} \left[\frac{x^4 y^2}{2} \right]_{-1}^{2} \, dx = \int_{x=0}^{1} \left[2x^4 - \frac{1}{2}x^4 \right] \, dx$$

$$= \int_0^1 \frac{3}{2}x^4 \, dx = \left[\frac{3}{10}x^5 \right]_0^1 = \frac{3}{10} \times 1 - \frac{3}{10} \times 0 = \frac{3}{10}$$

and

$$I = \int_{y=-1}^{2} \int_{x=0}^{1} x^4 y \;\; dxdy = \int_{y=-1}^{2} \left[\frac{x^5 y}{5} \right]_0^1 \, dy = \int_{-1}^{2} \left[\frac{y}{5} - 0 \right] \, dy$$

$$= \int_{-1}^{2} \frac{y}{5} \, dy = \left[\frac{y^2}{10} \right]_{-1}^{2} = \frac{4}{10} - \frac{1}{10} = \frac{3}{10}$$

Task

Evaluate the following integral:

$$I = \int_0^1 \int_{-1}^1 z\,(w+1)\;\; dwdz.$$

Your solution

Answer

1

Exercises

1. Evaluate the following integrals:

   (a) $I = \displaystyle\int_0^{\pi/2} \int_0^1 (y \cos x)\;\; dydx$

   (b) $I = \displaystyle\int_{-8}^3 \int_{-1}^1 y^2\;\; dydx$

   (c) $I = \displaystyle\int_0^1 \int_0^5 (s+1)^4\; dtds$

2. Evaluate the integrals $\displaystyle\int_{-1}^3 \int_0^2 x^3 y\;\;\; dydx$ and $\displaystyle\int_0^2 \int_{-1}^3 x^3 y\;\; dxdy$ and show that they are equal. As explained in the text, the order in which these integrations are carried out does not matter for integrations over rectangular areas.

Answers

1. (a) 1/2, (b) 22/3, (c) 31

2. 40

6. Applications of surface integration over rectangular areas

Force on a dam

At the beginning of this Section, the total force on a dam was given by the surface integral

$$\int_A k(h-y)\, dA$$

Imagine that the dam is rectangular in profile with a width of 100 m and a height h of 40 m. The expression dA is replaced by $dxdy$ and the limits on the variables x and y are 0 to 100 m and 0 to 40 m respectively. The constant k may be assumed to be 10^4 kg m^{-2} s^{-2}. The surface integral becomes the double integral

$$\int_0^{40}\int_0^{100} k(h-y)\;\; dxdy \qquad \text{that is} \qquad \int_0^{40}\int_0^{100} 10^4(40-y)\;\; dxdy$$

As the integral in this double integral does not contain x, the integral may be written

$$\begin{aligned}
\int_0^{40}\int_0^{100} 10^4(40-y)\;\; dxdy &= (100-0)\int_0^{40} 10^4(40-y)\, dy \\
&= 100\times 10^4\left[40y-\frac{y^2}{2}\right]_0^{40} \\
&= 10^6[(40\times 40-40^2/2)-0] \\
&= 10^6\times 800 = 8\times 10^8 \text{ N}
\end{aligned}$$

that is the total force is 800 meganewtons.

Centre of pressure

We wish to find the centre of pressure (x_p, y_p) of a plane area immersed vertically in a fluid. Take the x axis to be in the surface of the fluid and the y axis to be vertically down, so that the plane Oxy contains the area.

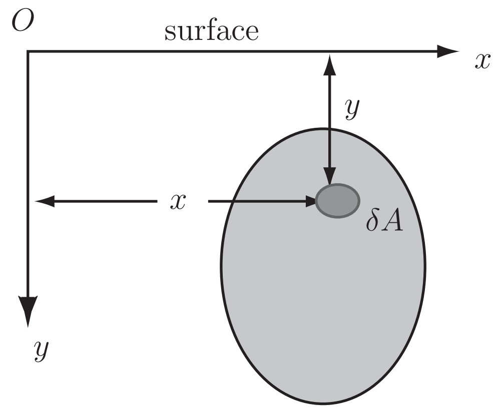


Figure 7

We require the following results:

(a) The pressure p is proportional to the depth h, so that $p=\omega h$ where ω is a constant.

(b) The force F on an area δA subjected to constant pressure p is given by $F=p\delta A$

Consider a small element of area δA at the position shown. The pressure at δA is ωy. Then the force acting on δA is $\omega y \delta A$. Hence the total force acting on the area A is $\displaystyle\int_A \omega y \; dA = \omega \int_A y \; dA$.

$$\begin{aligned}
\text{Moment of force on } \delta A \text{ about } Oy &= \omega x y \delta A \\
\text{Total moment of force on } \delta A \text{ about } Oy &= \omega \int_A xy \; dA \\
\text{Moment of force on } \delta A \text{ about } Ox &= \omega y^2 \delta A \\
\text{Total moment of force on } \delta A \text{ about } Ox &= \omega \int_A y^2 \; dA
\end{aligned}$$

Taking moments about Oy:

$$\begin{aligned}
\text{total force} \times x_p &= \text{total moment} \\
\left(\omega \int_A y \; dA\right) x_p &= \omega \int_A xy \; dA \\
x_p \int_A y \; dA &= \int_A xy \; dA
\end{aligned}$$

Taking moments about Ox:

$$\begin{aligned}
\text{total force} \times y_p &= \text{total moment} \\
\left(\omega \int_A y \; dA\right) y_p &= \omega \int_A y^2 \; dA \\
y_p \int_A y \; dA &= \int_A y^2 \; dA
\end{aligned}$$

Hence

$$x_p = \frac{\displaystyle\int_A xy \, dA}{\displaystyle\int_A y \, dA} \text{ and } y_p = \frac{\displaystyle\int_A y^2 \, dA}{\displaystyle\int_A y \, dA}.$$

Example 6

A rectangle of sides a and b is immersed vertically in a fluid with one of its edges in the surface as shown in Figure 8. Where is the centre of pressure?

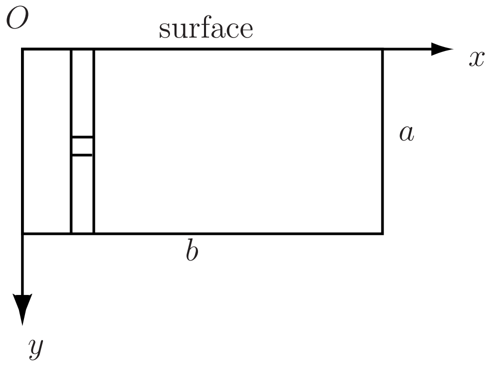


Figure 8

Solution

To express the surface integral as double integrals we will use cartesian coordinates and vertical slices. We need the following integrals.

$$\begin{aligned}
\int_A y \ dA &= \int_0^b \int_0^a y \ dydx = \int_0^b \left[\frac{1}{2}y^2\right]_0^a dx = \int_0^b \frac{1}{2}a^2 \ dx = \left[\frac{1}{2}a^2 x\right]_0^b = \frac{1}{2}a^2 b \\
\int_A xy \ dA &= \int_0^b \int_0^a xy \ dydx = \int_0^b \left[\frac{1}{2}xy^2\right]_0^a dx = \int_0^b \frac{1}{2}xa^2 \ dx = \left[\frac{1}{4}x^2 a^2\right]_0^b = \frac{1}{4}a^2 b^2 \\
\int_A y^2 \ dA &= \int_0^b \int_0^a y^2 \ dydx = \int_0^b \left[\frac{1}{3}y^3\right]_0^a dx = \int_0^b \frac{1}{3}a^3 \ dx = \left[\frac{1}{3}a^3 x\right]_0^b = \frac{1}{3}a^3 b
\end{aligned}$$

Hence

$$y_p = \frac{\displaystyle\int_A y^2 \ dA}{\displaystyle\int_A y \ dA} = \frac{\frac{1}{3}a^3 b}{\frac{1}{2}a^2 b} = \frac{2}{3}a \quad \text{and} \quad x_p = \frac{\displaystyle\int_A xy \ dA}{\displaystyle\int_A y \ dA} = \frac{\frac{1}{4}a^2 b^2}{\frac{1}{2}a^2 b} = \frac{1}{2}b$$

The centre of pressure is $(\frac{1}{2}b, \frac{2}{3}a)$, so is at a depth of $\frac{2}{3}a$.

Areas and moments

The surface integral $\int_A f(x,y)\ \ dA$ can represent a number of physical quantities, depending on the function $f(x,y)$ that is used.

Properties:

(a) If $f(x,y) = 1$ then the integral represents the area of A.

(b) If $f(x,y) = x$ then the integral represents the first moment of A about the y axis.

(c) If $f(x,y) = y$ then the integral represents the first moment of A about the x axis.

(d) If $f(x,y) = x^2$ then the integral represents the second moment of A about the y axis.

(e) If $f(x,y) = y^2$ then the integral represents the second moment of A about the x axis.

(f) If $f(x,y) = x^2 + y^2$ then the integral represents the second moment of A about the z axis.

Example 7

Given a rectangular lamina of length ℓ, width b, thickness t (small) and density ρ (see Figure 9), find the second moment of area of this lamina (moment of inertia) about the x-axis.

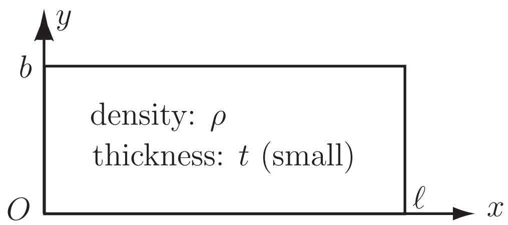


Figure 9

Solution

By property (e) above, the moment of inertia is given by

$$\begin{aligned}
\int_0^b \int_0^\ell y^2 \rho t\ \ dxdy &= \rho t(\ell - 0) \int_0^b y^2\ \ dy \\
&= \ell \rho t \left[\frac{y^3}{3}\right]_0^b \\
&= \ell \rho t \frac{b^3}{3}
\end{aligned}$$

As the mass of the lamina is $M = \ell b t \rho$, the moment of inertia simplifies to $\frac{1}{3}Mb^2$. The t and ρ are included in the integral to make it a moment of inertia rather than simply a second moment.

Task

By a similar method to that in Example 7, find the moment of inertia of the same lamina about the y-axis.

Your solution

Answer

From property (d) above, the moment of inertia (or second moment of area) is given by the integral

$$\begin{aligned}\int_0^l \int_0^b x^2 \rho t \; dydx &= \rho t(b-0)\int_0^l x^2 \; dx \\ &= b\rho t\left[\frac{x^3}{3}\right]_0^l \\ &= b\rho t\frac{l^3}{3}\end{aligned}$$

As the mass of the lamina is $M = lb\rho t$, the moment of inertia simplifies to $\frac{1}{3}Ml^2$. Again, the t and ρ are included in the integral to make it a moment of inertia rather than simple a second moment.

Exercises

By making use of the form of the integrand, evaluate the following double integrals:

1. $I = \displaystyle\int_0^\pi \int_0^1 y\cos^2 x \quad dydx$

2. $I = \displaystyle\int_{-8}^3 \int_{-1}^1 y^2 \;\; dydx$

3. $I = \displaystyle\int_0^1 \int_0^5 (s+1)^4 \, dtds$

Answers 1. $\frac{\pi}{4}$, 2. $\frac{22}{3}$ 3. 31

Multiple Integrals over Non-rectangular Regions

27.2

Introduction

In the previous Section we saw how to evaluate double integrals over simple rectangular regions. We now see how to extend this to non-rectangular regions.

In this Section we introduce functions as the limits of integration, these functions define the region over which the integration is performed. These regions can be non-rectangular. Extra care now must be taken when changing the order of integration. Producing a sketch of the region is often very helpful.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of two variables
- have completed Section 27.1
- be able to sketch a function in the plane

Learning Outcomes

On completion you should be able to ...

- evaluate double integrals over non-rectangular regions

1. Functions as limits of integration

In Section 27.1 double integrals of the form

$$I = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y)\ dydx$$

were considered. They represent an integral over a rectangular region in the xy plane. If the limits of integration of the inner integral are replaced with functions G_1, G_2,

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y)\ dydx$$

then the region described will not, in general, be a rectangle. The region will be a shape bounded by the curves (or lines) which these functions G_1 and G_2 describe.
As was indicated in 27.1

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y)dy\, dx$$

can be interpreted as the volume lying above the region in the xy plane defined by $G_1(x)$ and $G_2(x)$, bounded above by the surface $z = f(x, y)$. Not all double integrals are interpreted as volumes but this is often the case. If $z = f(x, y) < 0$ anywhere in the relevant region, then the double integral no longer represents a volume.

Key Point 4

Double Integral Over General Region

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y)\ dydx$$

1. The functions G_1, G_2 which are the limits for the inner integral are functions of the variable of the outer integral. This must be the case for the integral to make sense.

2. The limits of the outer integral are constant.

3. Integration over rectangular regions can be thought of as the special case where G_1 and G_2 are constant functions.

Example 8

Evaluate the integral $\quad I = \displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} 2xy\ dydx$

Solution

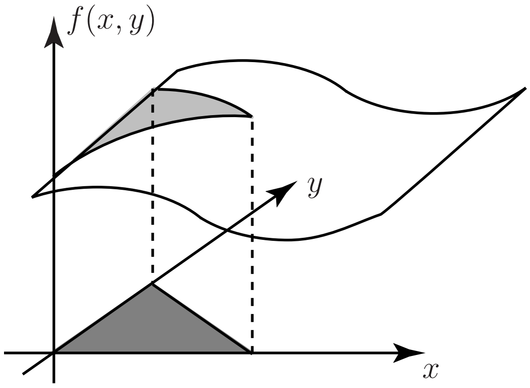


Figure 10

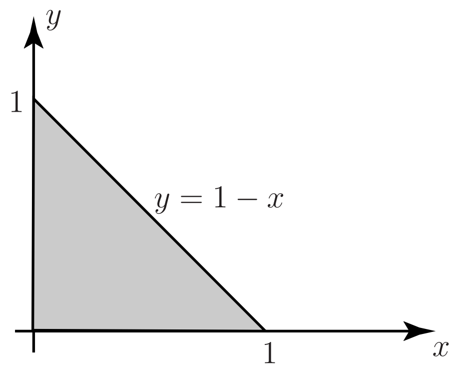


Figure 11

Projecting the relevant part of the surface (Figure 10) down to the xy plane produces the triangle shown in Figure 11. The extremes that x takes are $x = 0$ and $x = 1$ and so these are the limits on the outer integral. For any value of x, the variable y varies between $y = 0$ (at the bottom) and $y = 1 - x$ (at the top). Thus if the volume, shown in the diagram, under the function $f(x,y)$, bounded by this triangle is required then the following integral is to be calculated.

$$\int_{x=0}^{1}\int_{y=0}^{1-x} f\left(x,y\right)\ dydx$$

Once the correct limits have been determined, the integration is carried out in exactly the same manner as in Section 27.1

First consider the inner integral $\quad g(x) = \displaystyle\int_{y=0}^{1-x} 2xy\ dy$

Integrating $2xy$ with respect to y gives $xy^2 + C$ so $\quad g(x) = \left[xy^2\right]_{y=0}^{1-x} = x(1-x)^2$

Note that, as is required, this is a function of x, the variable of the outer integral. Now the outer integral is

$$\begin{aligned} I &= \int_{x=0}^{1} x(1-x)^2\ dx \\ &= \int_{x=0}^{1}\left(x^3 - 2x^2 + x\right)\ dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_{x=0}^{1} = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{1}{12} \end{aligned}$$

Regions do not have to be bounded only by straight lines. Also the integrals may involve other tools of integration, such as substitution or integration by parts. Drawing a sketch of the limit functions in the plane and shading the region is a valuable tools when evaluating such integrals.

Example 9

Evaluate the volume under the surface given by $z = f(x,y) = 2x\sin(y)$, over the region bounded above by the curve $y = x^2$ and below by the line $y = 0$, for $0 \leq x \leq 1$.

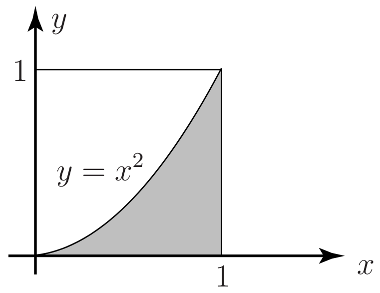


Figure 12

Solution

First sketch the curve $y = x^2$ and identify the region. This is the shaded region in Figure 12. The required integral is

$$\begin{aligned} I &= \int_{x=0}^{1}\int_{y=0}^{x^2} 2x\sin(y)\; dydx \\ &= \int_{x=0}^{1}\Big[-2x\cos(y)\Big]_{y=0}^{x^2}\; dx \\ &= \int_{x=0}^{1}\left(-2x\cos\left(x^2\right)+2x\right)\; dx \\ &= \int_{x=0}^{1}\left(1-\cos\left(x^2\right)\right)2x\;\; dx \end{aligned}$$

Making the substitution $u = x^2$ so $du = 2x\; dx$ and noting that the limits $x = 0, 1$ map to $u = 0, 1$, gives

$$\begin{aligned} I &= \int_{u=0}^{1}\left(1-\cos\left(u\right)\right)du \\ &= \Big[u-\sin(u)\Big]_{u=0}^{1} \\ &= 1-\sin(1) \\ &\approx 0.1585 \end{aligned}$$

Example 10

Evaluate the volume under the surface given by $z = f(x, y) = x^2 + \frac{1}{2}y$, over the region bounded by the curves $y = 2x$ and $y = x^2$.

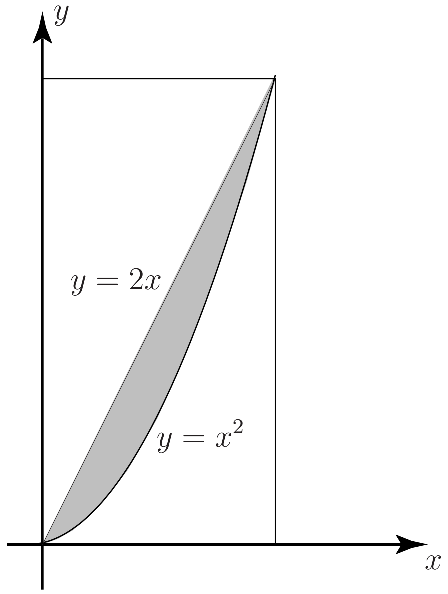


Figure 13

Solution

The sketch of the region is shown in Figure 13. The required integral is

$$I = \int_{x=a}^{b} \int_{y=x^2}^{2x} \left(x^2 + \frac{1}{2}y \right) \, dydx$$

To determine the limits for the integration with respect to x, the points where the curves intersect are required. These points are the solutions of the equation $2x = x^2$, so the required limits are $x = 0$ and $x = 2$. Then the volume is given by

$$\begin{aligned} I &= \int_{x=0}^{2} \int_{y=x^2}^{2x} \left(x^2 + \frac{1}{2}y \right) \, dydx \\ &= \int_{x=0}^{2} \left[x^2 y + \frac{1}{4}y^2 \right]_{y=x^2}^{2x} \, dx \\ &= \int_{x=0}^{2} \left(x^2 + 2x^3 - \frac{5}{4}x^4 \right) \, dx \\ &= \left[\frac{x^3}{3} + \frac{x^4}{2} - \frac{x^5}{4} \right]_{x=0}^{2} \\ &= \frac{8}{3} \end{aligned}$$

Example 11

(a) Evaluate the volume under $z = f(x, y) = 5x^2y$, over the half of the unit circle that lies above the x-axis. (Figure 14).

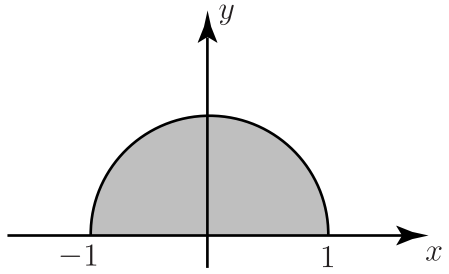


Figure 14

(b) Repeat (a) for $z = f(x, y) = 1$.

Solution

(a) This region is bounded by the circle $y^2 + x^2 = 1$ and the line $y = 0$. Since only positive values of y are required, the equation of the circle can be written $y = \sqrt{1 - x^2}$. Then the required volume is given by

$$\begin{aligned} I &= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(5x^2y\right)\, dydx \quad = \int_{x=-1}^{1} \left[\frac{5}{2}x^2y^2\right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{x=-1}^{1} \frac{5}{2}x^2(1-x^2)\, dx \quad = \frac{5}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{3} \end{aligned}$$

(b)

$$\begin{aligned} I &= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} 1\, dydx \quad = \int_{x=-1}^{1} \Big[y\Big]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{x=-1}^{1} \sqrt{1-x^2}\, dx \text{ (which by substituting } x = \sin\theta) = \frac{\pi}{2} \end{aligned}$$

Note that by putting $f(x, y) = 1$ we have found the volume of a semi-circular lamina of uniform height 1. This result is numerically the same as the **area** of the region in Figure 14. (This is a general result.)

Task

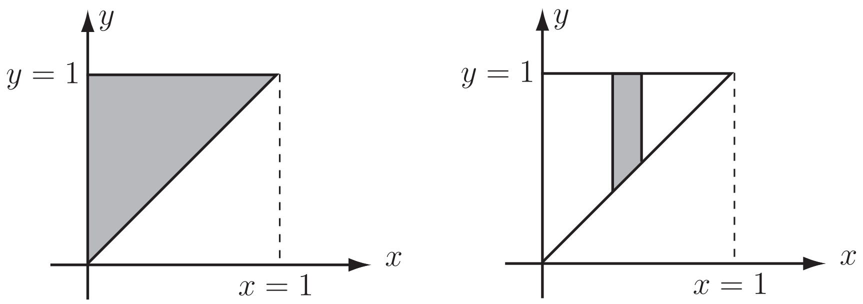
Evaluate the following double integral over a non-rectangular region.

$$\int_{x=0}^{1} \int_{y=x}^{1} \left(x^2 + y^2\right)\ dydx$$

(a) First sketch the region of the xy-plane determined by the limits:

Your solution

Answer



(b) Now evaluate the inner triangle:

Your solution

Answer

In the triangle, x varies between $x = 0$ and $x = 1$. For every value of x, y varies between $y = x$ and $y = 1$.

The inner integral is given by

$$\begin{aligned}
\text{Inner Integral} = \int_{y=x}^{1} \left(x^2 + y^2\right) dy \quad &= \quad \left[x^2 y + \frac{1}{3} y^3\right]_x^1 \\
&= \quad x^2 \times 1 + \frac{1}{3} \times 1^3 - (x^2 \times x + \frac{1}{3} x^3) \\
&= \quad x^2 + \frac{1}{3} - \frac{4}{3} x^3
\end{aligned}$$

(c) Finally evaluate the outer integral:

Your solution

Answer

The inner integral is placed in the outer integral to give

$$\text{Outer Integral} = \int_0^1 \left(x^2 - \frac{4}{3}x^3 + \frac{1}{3}\right)\, dx = \left[\frac{1}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{3}x\right]_0^1$$
$$= (\frac{1}{3} - \frac{1}{3} + \frac{1}{3}) - 0$$
$$= \frac{1}{3}$$

Note that the above Task is simply one of integrating a function over a region - there is no reference to a volume here. Another like this now follows.

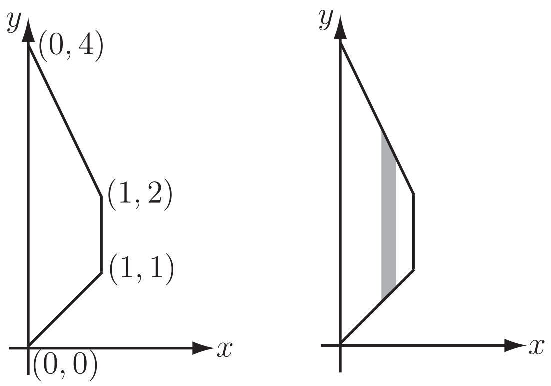
Task

Integrate the function $z = x^2 y$ over the trapezium with vertices at $(0, 0)$, $(1, 1)$, $(1, 2)$ and $(0, 4)$.

Your solution

Answer

The integration takes place over the trapezium shown (left)



Considering variable x on the outer integral and variable y on the inner integral, the trapezium has an extent in x of $x = 0$ to $x = 1$. So, the limits on the outer integral (limits on x) are $x = 0$ and $x = 1$.

For each value of x, y varies from $y = x$ (line joining $(0, 0)$ to $(1, 1)$) to $y = 4 - 2x$ (line joining $(1, 2)$ and $(0, 4)$). So the limits on the inner integral (limits on y) are $y = x$ to $y = 4 - 2x$.

The double integral thus becomes

$$\int_{x=0}^{1}\int_{y=x}^{4-2x} x^2y\ dy\ dx$$

The inner integral is

$$\int_{y=x}^{4-2x} x^2y\ dy = \left[x^2\frac{y^2}{2}\right]_{y=x}^{4-2x} = x^2\frac{(4-2x)^2}{2} - x^2\frac{x^2}{2} = 8x^2 - 8x^3 + \frac{3}{2}x^4$$

Putting this into the outer integral gives

$$\int_{x=0}^{1}(8x^2 - 8x^3 + \frac{3}{2}x^4)\ dx = \left[\frac{8}{3}x^3 - 2x^4 + \frac{3}{10}x^5\right]_0^1 = (\frac{8}{3} - 2 + \frac{3}{10}) - 0 = \frac{29}{30}$$

Exercises

Evaluate the following integrals

1. $\displaystyle\int_{x=0}^{1}\int_{y=3x}^{x^2+2} xy\ dydx$

2. $\displaystyle\int_{x=1}^{2}\int_{y=x^2+2}^{3x} xy\ dydx$ [Hint: Note how the same curves can define different regions.]

3. $\displaystyle\int_{x=1}^{2}\int_{y=1}^{x^2} \frac{x}{y}\ dydx,$ [Hint: use integration by parts for the outer integral.]

Answers

1. $\dfrac{11}{24}$

2. $\dfrac{9}{8}$

3. $4\ln 2 - \dfrac{3}{2} \approx 1.27$

Splitting the region of integration

Sometimes it is difficult or impossible to represent the region of integration by means of consistent limits on x and y. Instead, it is possible to divide the region of integration into two (or more) sub-regions, carry out a multiple integral on each region and add the integrals together. For example, suppose it is necessary to integrate the function $g(x, y)$ over the triangle defined by the three points $(0, 0)$, $(1, 4)$ and $(2, -2)$.

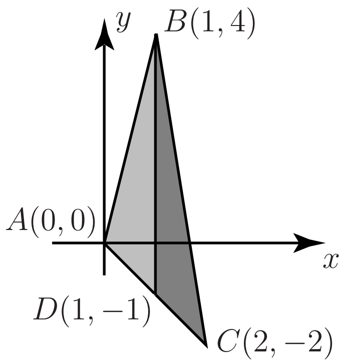


Figure 15

It is not possible to represent the triangle ABC by means of limits on an inner integral and an outer integral. However, it can be split into the triangle ABD and the triangle BCD. D is chosen to be the point on AC directly beneath B, that is, line BD is parallel to the y-axis so that x is constant along it. Note that the sides of triangle ABC are defined by sections of the lines $y = 4x$, $y = -x$ and $y = -6x + 10$.

In triangle ABD, the variable x takes values between $x = 0$ and $x = 1$. For each value of x, y can take values between $y = -x$ (bottom) and $y = 4x$. Hence, the integral of the function $g(x, y)$ over triangle ABD is

$$I_1 = \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} g\left(x, y\right)\ dydx$$

Similarly, the integral of $g(x, y)$ over triangle BCD is

$$I_2 = \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} g\left(x, y\right)\ dydx$$

and the integral over the full triangle is

$$I = I_1 + I_2 = \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} g\left(x, y\right)\ dydx + \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} g\left(x, y\right)\ dydx$$

Example 12

Integrate the function $g(x, y) = xy$ over the triangle ABC.

Solution

Over triangle ABD, the integral is

$$\begin{aligned}
I_1 &= \int_{x=0}^{x=1} \int_{y=-x}^{y=4x} xy \; dydx \\
&= \int_{x=0}^{x=1} \left[\frac{1}{2} xy^2 \right]_{y=-x}^{y=4x} dx = \int_{x=0}^{x=1} \left[8x^3 - \frac{1}{2}x^3 \right] \; dx \\
&= \int_{x=0}^{x=1} \frac{15}{2} x^3 \; dx = \left[\frac{15}{8} x^4 \right]_0^1 = \frac{15}{8} - 0 = \frac{15}{8}
\end{aligned}$$

Over triangle BCD, the integral is

$$\begin{aligned}
I_2 &= \int_{x=1}^{x=2} \int_{y=-x}^{y=-6x+10} xy \; dydx \\
&= \int_{x=1}^{x=2} \left[\frac{1}{2} xy^2 \right]_{y=-x}^{y=-6x+10} dx = \int_{x=1}^{x=2} \left[\frac{1}{2} x(-6x+10)^2 - \frac{1}{2} x(-x)^2 \right] \; dx \\
&= \int_{x=1}^{x=2} \frac{1}{2} \left[36x^3 - 120x^2 + 100x - x^3 \right] \; dx = \frac{1}{2} \int_{x=1}^{x=2} \left[35x^3 - 120x^2 + 100x \right] \; dx \\
&= \frac{1}{2} \left[\frac{35}{4} x^4 - 40x^3 + 50x^2 \right]_1^2 = 10 - \frac{75}{8} = \frac{5}{8}
\end{aligned}$$

So the total integral is $I_1 + I_2 = \dfrac{15}{8} + \dfrac{5}{8} = \dfrac{5}{2}$

2. Order of integration

All of the preceding Examples and Tasks have been integrals of the form

$$I = \int_{x=a}^{x=b} \int_{G_1(x)}^{G_2(x)} f(x, y) \; dydx$$

These integrals represent taking vertical slices through the volume that are parallel to the yz-plane. That is, vertically through the xy-plane.

Just as for integration over rectangular regions, the order of integration can be changed and the region can be sliced parallel to the xz-plane. If the inner integral is taken with respect to x then an integral of the following form is obtained:

$$I = \int_{y=c}^{y=d} \int_{H_1(y)}^{H_2(y)} f(x, y) \; dxdy$$

Key Point 5

Changing Order of Integration

1. The integrand $f(x, y)$ is not altered by changing the order of integration.
2. The limits will, in general, be different.

Example 13

The following integral was evaluated in Example 9.

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} 2x \sin(y)\, dydx = 1 - \sin(1)$$

Change the order of integration and confirm that the new integral gives the same result.

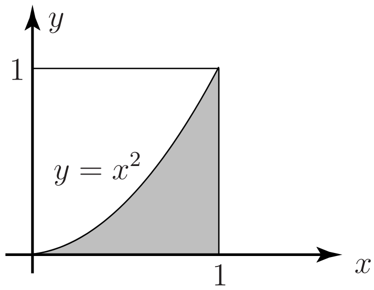


Figure 16

Solution

The integral is taken over the region which is bounded by the curve $y = x^2$. Expressed as a function of y this curve is $x = \sqrt{y}$. Now consider this curve as bounding the region from the left, then the line $x = 1$ bounds the region to the right. These then are the limit functions for the inner integral $H_1(y) = \sqrt{y}$ and $H_2(y) = 1$. Then the limits for the outer integral are $c = 0 \leq y \leq 1 = d$. The following integral is obtained

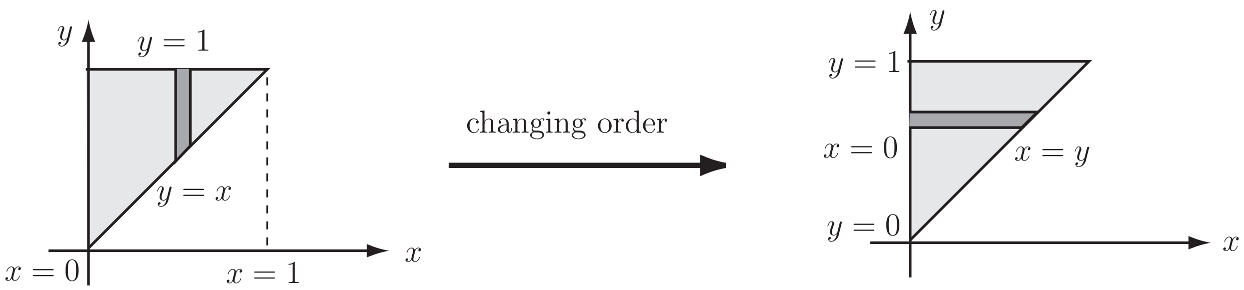
$$\begin{aligned} I &= \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} 2x \sin(y)\ dxdy = \int_{y=0}^{1} \left[x^2 \sin(y) \right]_{x=\sqrt{y}}^{x=1} dy = \int_{y=0}^{1} (1 - y) \sin(y))\, dy \\ &= \left[-(1 - y) \cos(y) \right]_{y=0}^{1} - \int_{y=0}^{1} \cos(y)\, dy, \quad \text{using integration by parts} \\ &= 1 - \left[\sin(y) \right]_{y=0}^{1} = 1 - \sin(1) \end{aligned}$$

The double integral $I = \int_0^1 \int_x^1 \mathrm{e}^{y^2}\ dydx$ involves an inner integral which is impossible to integrate. Show that if the order of integration is reversed, the integral can be expressed as $I = \int_0^1 \int_0^y \mathrm{e}^{y^2}\ dxdy$. Hence evaluate the integral I.

Your solution

Answer

The following diagram shows the changing description of the boundary as the order of integration is changed.



$$I = \int_0^1 \int_x^1 \mathrm{e}^{y^2}\ dydx = \int_0^1 \int_0^y \mathrm{e}^{y^2} dxdy = \int_0^1 \left[x\mathrm{e}^{y^2} \right]_0^y dy$$

$$= \int_0^1 y\mathrm{e}^{y^2} dy = \left[\frac{1}{2}\mathrm{e}^{y^2} \right]_0^1 = \frac{1}{2}(\mathrm{e} - 1)$$

3. Evaluating surface integrals using polar coordinates

Areas with circular boundaries often lead to double integrals with awkward limits, and these integrals can be difficult to evaluate. In such cases it is easier to work with polar (r, θ) rather than Cartesian (x, y) coordinates.

Polar coordinates

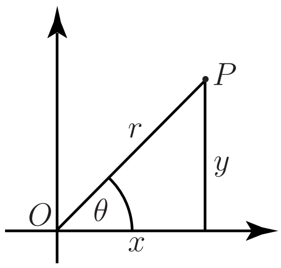


Figure 17

The polar coordinates of the point P are the distance r from P to the origin O and the angle θ that the line OP makes with the positive x axis. The following are used to transform between polar and rectangular coordinates.

1. Given (x, y), (r, θ) are found using $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
2. Given (r, θ), (x, y) are found using $x = r\cos\theta$ and $y = r\sin\theta$

Note that we also have the relation $r^2 = x^2 + y^2$.

Finding surface integrals with polar coordinates

The area of integration A is covered with coordinate circles given by $r =$ constant and coordinate lines given by $\theta =$ constant.
The elementary areas δA are almost rectangles having width δr and length determined by the length of the part of the circle of radius r between θ and $\delta\theta$, the arc length of this part of the circle is $r\delta\theta$. So $\delta A \approx r\delta r\delta\theta$. Thus to evaluate $\int_A f(x, y)\, dA$ we sum $f(r, \theta) r\delta r\delta\theta$ for all δA.

$$\int_A f(x, y)\, dA = \int_{\theta=\theta_A}^{\theta=\theta_B} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r, \theta)\ r\ drd\theta$$

Key Point 6

Polar Coordinates

In double integration using polar coordinates, the variable r appears in $f(r, \theta)$ and in $rdrd\theta$. As explained above, this r is required because the elementary area element become larger further away from the origin.

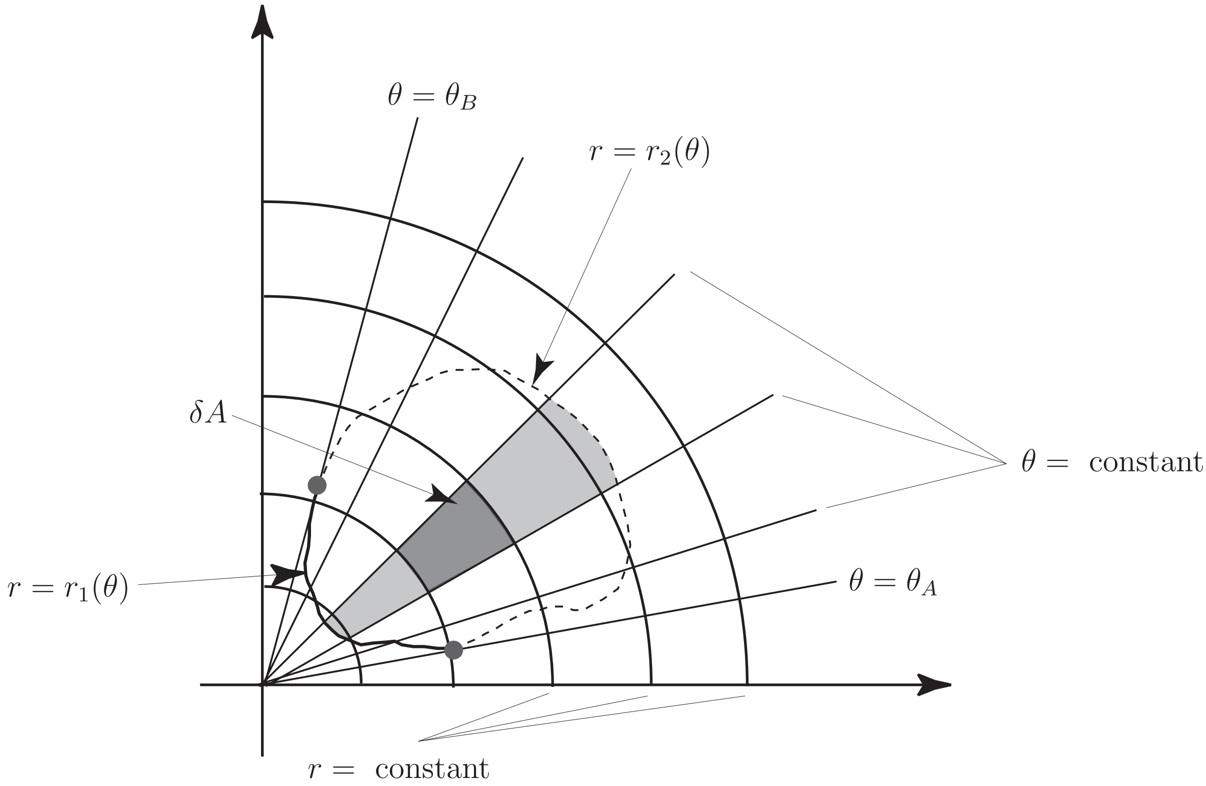


Figure 18

Note that the use of polar coordinates is a special case of the use of a change of variables. Further cases of change of variables will be considered in Section 27.4.

Example 14

Evaluate $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^2 r\cos\theta\, drd\theta$ and sketch the region of integration. Note that it is the function $\cos\theta$ which is being integrated over the region and the r comes from the $rdrd\theta$.

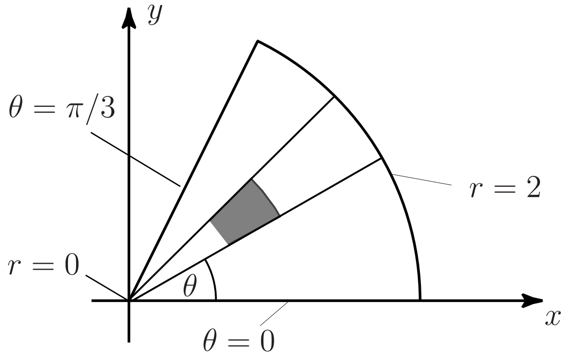


Figure 19

Solution

The evaluation is similar to that for cartesian coordinates. The inner integral with respect to r, is evaluated first with θ constant. Then the outer θ integral is evaluated.

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\int_0^2 r\cos\theta\;d\theta &= \int_0^{\frac{\pi}{3}}\left[\frac{1}{2}r^2\cos\theta\right]_0^2 d\theta \\ &= \int_0^{\frac{\pi}{3}} 2\cos\theta\;d\theta \\ &= \Big[2\sin\theta\Big]_0^{\frac{\pi}{3}} = 2\sin\frac{\pi}{3} = \sqrt{3}\end{aligned}$$

With θ constant r varies between 0 and 2, so the bounding curves of the polar strip start at $r = 0$ and end at $r = 2$. As θ varies between 0 and $\frac{\pi}{3}$ a sector of a circular disc is swept out. This sector is the region of integration shown above.

Example 15

Earlier in this Section, an example concerned integrating the function $f(x, y) = 5x^2y$ over the half of the unit circle which lies above the x-axis. It is also possible to carry out this integration using polar coordinates.

Solution

The semi-circle is characterised by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. So the integral may be written (remembering that $x = r\cos\theta$ and $y = r\sin\theta$)

$$\int_0^{\pi}\int_0^1 5(r\cos\theta)^2(r\sin\theta)\;r\;drd\theta$$

which can be evaluated as follows

$$\begin{aligned}&\int_0^{\pi}\int_0^1 5r^4\sin\theta\cos^2\theta\;drd\theta \\ =&\int_0^{\pi}\left[r^5\sin\theta\cos^2\theta\right]_0^1 d\theta \\ =&\int_0^{\pi}\sin\theta\cos^2\theta\;d\theta = \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi} = -\frac{1}{3}\cos^3\pi + \frac{1}{3}\cos^3 0 = -\frac{1}{3}(-1) + \frac{1}{3}(1) = \frac{2}{3}\end{aligned}$$

This is, of course, the same answer that was obtained using an integration over rectangular coordinates.

4. Applications of surface integration

Force on a dam

Section 27.1 considered the force on a rectangular dam of width 100 m and height 40 m. Instead, imagine that the dam is not rectangular in profile but instead has a width of 100 m at the top but only 80 m at the bottom. The top and bottom of the dam can be given by line segments $y = 0$ (bottom) and $y = 40$ while the sides are parts of the lines $y = 40 - 4x$ i.e. $x = 10 - \dfrac{y}{4}$ (left) and $y = 40 + 4(x - 100) = 4x - 360$ i.e. $x = 90 + \dfrac{y}{4}$(right). (See Figure 20).

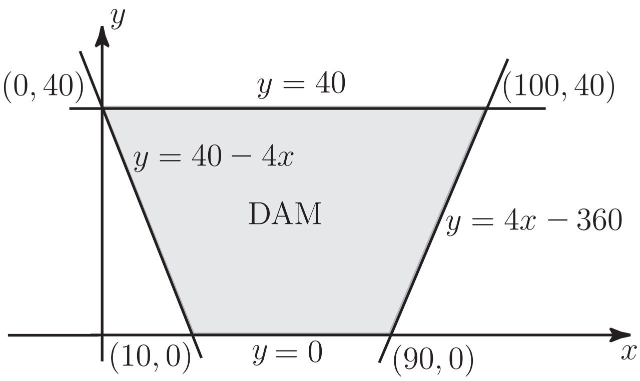


Figure 20

Thus the dam exists at heights y between 0 and 40 while for each value of y, the horizontal coordinate x varies between $x = 10 - \dfrac{y}{4}$ and $x = 90 + \dfrac{y}{4}$. Thus the surface integral representing the total force i.e.

$I = \displaystyle\int_A 10^4(40 - y)\, dA$ becomes the double integral $I = \displaystyle\int_0^{40} \int_{10-\frac{y}{4}}^{90+\frac{y}{4}} 10^4(40 - y)\ \ dxdy$

which can be evaluated as follows

$$
\begin{aligned}
I &= \int_0^{40} \int_{10-\frac{y}{4}}^{90+\frac{y}{4}} 10^4(40 - y)\ \ dxdy \\
&= 10^4 \int_0^{40} \Big[\ \ (40 - y)x\Big]_{10-\frac{y}{4}}^{90+\frac{y}{4}}\ dy = 10^4 \int_0^{40} \left[(40 - y)(90 + \frac{y}{4}) - (40 - y)(10 - \frac{y}{4})\right]\ dy \\
&= 10^4 \int_0^{40} \left[(40 - y)(80 + \frac{y}{2})\right]\ dy = 10^4 \int_0^{40} \left[3200 - 60y - \frac{y^2}{2}\right]\ dy \\
&= 10^4 \left[3200y - 30y^2 - \frac{1}{6}y^3\right]_0^{40} = 10^4 \left[(3200 \times 40 - 30 \times 40^2 - \frac{1}{6}40^3) - 0\right] \\
&= 10^4 \times \frac{208000}{3} \approx 6.93 \times 10^8 \text{N}
\end{aligned}
$$

i.e. the total force is just under 700 meganewtons.

Centre of pressure

A plane area in the shape of a quadrant of a circle of radius a is immersed vertically in a fluid with one bounding radius in the surface. Find the position of the centre of pressure.

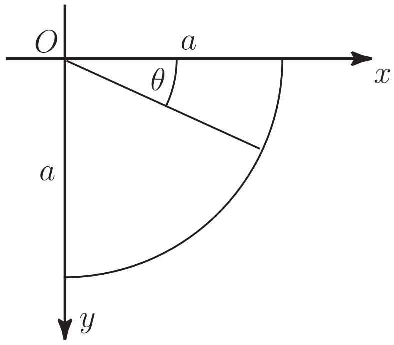


Figure 21

Note: In subsection 6 of Section 27.1 it was shown that the coordinates of the centre of pressure of a (thin) object are

$$x_p = \frac{\displaystyle\int_A xy\;dA}{\displaystyle\int_A y\;dA} \qquad \text{and} \qquad y_p = \frac{\displaystyle\int_A y^2\;dA}{\displaystyle\int_A y\;dA}$$

$$\begin{aligned}
\int_A y\;dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^2\sin\theta\;drd\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{3}r^3\sin\theta\right]_0^a\;d\theta \\
&= \frac{1}{3}a^3\int_0^{\frac{\pi}{2}}\sin\theta\;d\theta = \frac{1}{3}a^3\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{1}{3}a^3 \\
\int_A xy\;dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^3\cos\theta\sin\theta\;d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_0^a\;d\theta \\
&= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\;d\theta = \frac{1}{4}a^4\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}a^4 \\
\int_A y^2\;dA &= \int_0^{\frac{\pi}{2}}\int_0^a r^3\sin^2\theta\;drd\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\sin^2\theta\right]_0^a\;d\theta \\
&= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\;d\theta = \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(1-\cos 2\theta\right)\;d\theta = \frac{1}{8}a^4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{16}\pi a^4
\end{aligned}$$

Then $x_p = \dfrac{\displaystyle\int_A xy\;dA}{\displaystyle\int_A y\;dA} = \dfrac{\frac{1}{8}a^4}{\frac{1}{3}a^3} = \dfrac{3}{8}a$ and $y_p = \dfrac{\displaystyle\int_A y^2\;dA}{\displaystyle\int_A y\;dA} = \dfrac{\frac{1}{16}\pi a^4}{\frac{1}{3}a^3} = \dfrac{3}{16}\pi a.$

The centre of pressure is at $\left(\dfrac{3}{8}a, \dfrac{3}{16}\pi a\right)$.

Engineering Example 1

Volume of liquid in an elliptic tank

Introduction

A tank in the shape of an elliptic cylinder has a volume of liquid poured into it. It is useful to know in advance how deep the liquid will be. In order to make this calculation, it is necessary to perform a multiple integration.

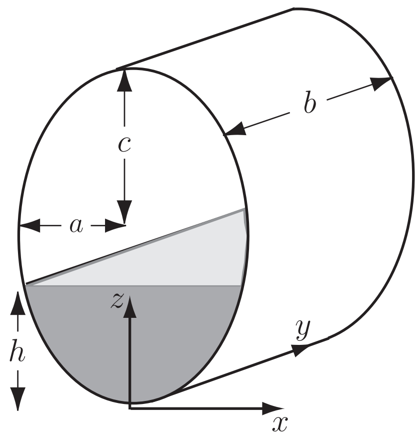


Figure 22

Problem in words

The tank has semi-axes a (horizontal) and c (vertical) and is of constant thickness b. A volume of liquid V is poured in (assuming that $V < \pi abc$, the volume of the tank), filling it to a depth h, which is to be calculated. Assume 3-D coordinate axes based on a point at the bottom of the tank.

Mathematical statement of the problem

Since the tank is of constant thickness b, the volume of liquid is given by the shaded area multiplied by b, i.e.

$$V = b \times \text{shaded area}$$

where the shaded area can be expressed as the double integral

$$\int_{z=0}^{h} \int_{x=x_1}^{x_2} dxdz$$

where the limits x_1 and x_2 on x can be found from the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{(z-c)^2}{c^2} = 1$$

Mathematical analysis

From the equation of the ellipse

$$
\begin{aligned}
x^2 &= a^2\left[1-\frac{(z-c)^2}{c^2}\right] \\
&= \frac{a^2}{c^2}\left[c^2-(z-c)^2\right] \\
&= \frac{a^2}{c^2}\left[2zc-z^2\right] \qquad \text{so} \quad x=\pm\frac{a}{c}\sqrt{2zc-z^2}
\end{aligned}
$$

Thus

$$x_1=-\frac{a}{c}\sqrt{2zc-z^2}, \quad \text{and} \quad x_2=+\frac{a}{c}\sqrt{2zc-z^2}$$

Consequently

$$
\begin{aligned}
V=b\int_{z=0}^{h}\int_{x=x_1}^{x_2} dxdz &= b\int_{z=0}^{h}\Big[x\Big]_{x_1}^{x_2} dz \\
&= b\int_{z=0}^{h} 2\frac{a}{c}\sqrt{2zc-z^2}\, dz
\end{aligned}
$$

Now use substitution $z-c=c\,\sin\theta$ so that $dz=c\,\cos\theta\,d\theta$

$$
\begin{aligned}
&z=0 \quad \text{gives} \quad \theta=-\frac{\pi}{2} \\
&z=h \quad \text{gives} \quad \theta=\sin^{-1}\left(\frac{h}{c}-1\right)=\theta_0 \quad \text{(say)}
\end{aligned}
$$

$$
\begin{aligned}
V &= b\int_{-\frac{\pi}{2}}^{\theta_0} 2\frac{a}{c}\; c\cos\theta\; c\cos\theta\; d\theta \\
&= 2abc\int_{-\frac{\pi}{2}}^{\theta_0} \cos^2\theta\; d\theta \\
&= abc\int_{-\frac{\pi}{2}}^{\theta_0} \left[1+\cos 2\theta\right]\; d\theta \\
&= abc\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\theta_0} \\
&= abc\left[\theta_0+\frac{1}{2}\sin 2\theta_0-\left(-\frac{\pi}{2}+0\right)\right] \\
&= abc\left[\theta_0+\frac{1}{2}\sin 2\theta_0+\frac{\pi}{2}\right] \qquad \ldots(*)
\end{aligned}
$$

which can also be expressed in the form

$$V=abc\left[\sin^{-1}\left(\frac{h}{c}-1\right)+\left(\frac{h}{c}-1\right)\sqrt{1-\left(\frac{h}{c}-1\right)^2}+\frac{\pi}{2}\right]$$

While $(*)$ expresses V as a function of θ_0 (and therefore h) to find θ_0 as a function of V requires a numerical method. For a given a, b, c and V, solve equation $(*)$ by a numerical method to find θ_0 and find h from $h = c(1 + \sin\theta_0)$.

Interpretation

If $a = 2$ m, $b = 1$ m, $c = 3$ m (so the total volume of the tank is 6π m$^3 \approx 18.85$ m^3), and a volume of 7 m^3 is to be poured into the tank then

$$V = abc\left[\theta_0 + \frac{1}{2}\sin 2\theta_0 + \frac{\pi}{2}\right]$$

which becomes

$$7 = 6\left[\theta_0 + \frac{1}{2}\sin 2\theta_0 + \frac{\pi}{2}\right]$$

and has solution $\theta_0 = -0.205$ (3 decimal places).

Finally

$$\begin{aligned} h &= c(1 + \sin\theta_0) \\ &= 3(1 + \sin(-0.205)) \\ &= 2.39 \text{ m to 2 d.p} \end{aligned}$$

compared to the maximum height of 6 m.

Exercises

1. Evaluate the functions (a) xy and (b) $xy + 3y^2$

   over the quadrilateral with vertices at $(0,0)$, $(3,0)$, $(2,2)$ and $(0,4)$.

2. Show that $\displaystyle\int\int_A f(x,y)\,dy\,dx = \int\int_A f(x,y)\,dx\,dy$ for $f(x,y) = xy^2$ when A is the interior of the triangle with vertices at $(0,0)$, $(2,0)$ and $(2,4)$.

3. By reversing the order of the two integrals, evaluate the integral $\displaystyle\int_{y=0}^{4}\int_{x=y^{1/2}}^{2} \sin x^3\,dx\,dy$

4. Integrate the function $f(x,y) = x^3 + xy^2$ over the quadrant $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$.

Answers

1. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{4-x} f(x,y)\,dy\,dx + \int_{x=2}^{3}\int_{y=0}^{6-2x} f(x,y)\,dy\,dx;\qquad \frac{22}{3} + \frac{3}{2} = \frac{53}{6}\ ;\ \frac{202}{3} + \frac{7}{2} = \frac{425}{6}$

2. Both equal $\dfrac{256}{15}$

3. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x^2} \sin x^3\,dy\,dx = \frac{1}{3}(1 - \cos 8) \approx 0.382$

4. $\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^4\cos\theta\,dr\,d\theta = \frac{1}{5}$

Volume Integrals 27.3

Introduction

In the previous two Sections, surface integrals (or double integrals) were introduced i.e. functions were integrated with respect to one variable and then with respect to another variable. It is often useful in engineering to extend the process to an integration with respect to three variables i.e. a volume integral or triple integral. Many of the processes and techniques involved in double integration are relevant to triple integration.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of two variables
- have studied Sections 27.1 and 27.2 on double integration
- be able to visualise or sketch a function in three variables.

Learning Outcomes

On completion you should be able to ...

- evaluate triple integrals

1. Example of volume integral: mass of water in a reservoir

Sections 27.1 and 27.2 introduced an example showing how the force on a dam can be represented by a double integral. Suppose, instead of the total force on the dam, an engineer wishes to find the total mass of water in the reservoir behind the dam. The mass of a little element of water (dimensions δx in length, δy in breadth and δz in height) with density ρ is given by $\rho\delta z\delta y\delta x$ (i.e. the mass of the element is given by its density multiplied by its volume).

The density may vary at different parts of the reservoir e.g. due to temperature variations and the water expanding at higher temperatures. It is important to realise that the density ρ may be a function of all three variables, x, y and z. For example, during the spring months, the depths of the reservoir may be at the cold temperatures of the winter while the parts of the reservoir nearer the surface may be at higher temperatures representing the fact that they have been influenced by the warmer air above; this represents the temperature varying with the vertical coordinate z. Also, the parts of the reservoir near where streams flow in may be extremely cold as melting snow flows into the reservoir. This represents the density varying with the horizontal coordinates x and y.

Thus the mass of a small element of water is given by $\rho(x, y, z)\delta z\delta y\delta x$ The mass of water in a column is given by the integral $\displaystyle\int_{-h(x,y)}^{0} \rho(x, y, z)\ dz\delta y\delta x$ where the level $z = 0$ represents the surface of the reservoir and the function $h(x, y)$ represents the depth of the reservoir for the particular values of x and y under consideration. [Note that the depth is positive but as it is measured downwards, it represents a negative value of z.]

The mass of water in a slice (aligned parallel to the x-axis) is given by integrating once more with respect to y i.e. $\displaystyle\int_{y_1(x)}^{y_2(x)}\int_{-h(x,y)}^{0} \rho(x, y, z)\ dzdy\delta x$. Here the functions $y_1(x)$ and $y_2(x)$ represent the extreme values of y for the value of x under consideration.

Finally the total mass of water in the reservoir can be found by integrating over all x i.e.

$$\int_a^b\int_{y_1(x)}^{y_2(x)}\int_{-h(x,y)}^{0} \rho(x, y, z)\ dzdydx.$$

To find the total mass of water, it is necessary to integrate the density three times, firstly with respect to z (between limits dependent on x and y), then with respect to y (between limits which are functions of x) and finally with respect to x (between limits which are constant).

This is an example of a triple or volume integral.

2. Evaluating triple integrals

A triple integral is an integral of the form

$$\int_a^b\int_{p(x)}^{q(x)}\int_{r(x,y)}^{s(x,y)} f(x, y, z)\ dzdydx$$

The evaluation can be split into an "inner integral" (the integral with respect to z between limits which are functions of x and y), an "intermediate integral" (the integration with respect to y between limits which are functions of x) and an "outer integral" (the integration with respect to x between

limits which are constants. Note that there is nothing special about the variable names x, y and z: other variable names could have been used instead.
Triple integrals can be represented in different ways. $\int_V f\, dV$ represents a triple integral where the dV is replaced by $dxdydz$ (or equivalent) and the limit of V on the integral is replaced by appropriate limits on the three integrals.
Note that the integral $\int_V dV$ (i.e. integrating the function $f(x,y,z)=1$) gives the volume of the relevant shape. Hence the alternative name of volume integral.
One special case is where the limits on **all** the integrals are constants (a constant is, of course, a special case of a function). This represents an integral over a cuboidal region.

Example 16

Consider a cube V of side 1.

(a) Express the integral $\int_V f\, dV$ (where f is any function of x, y and z) as a triple integral.

(b) Hence evaluate $\int_V (y^2+z^2)\, dV$

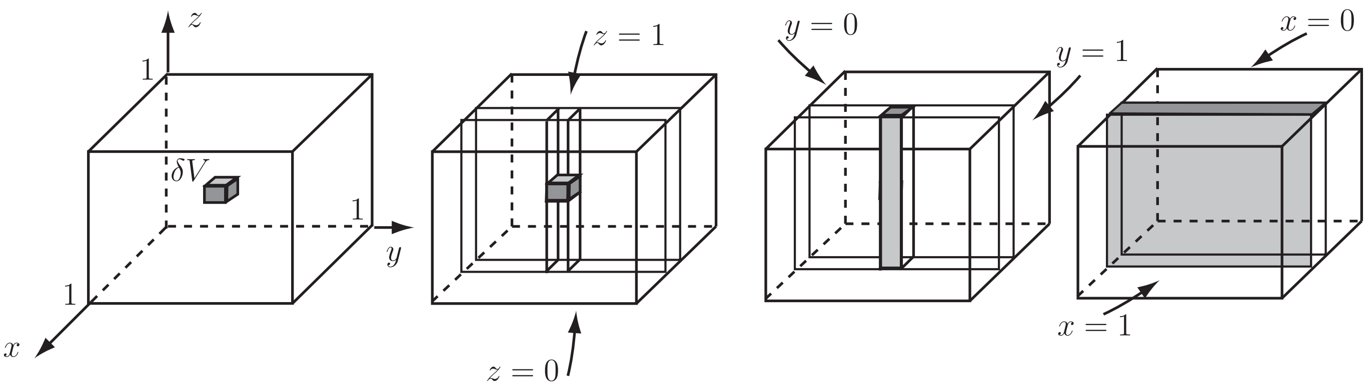


Figure 23

Solution

(a) Consider a little element of length dx, width dy and height dz. Then δV (the volume of the small element) is the product of these lengths $dxdydz$. The function is integrated three times. The first integration represents the integral over the vertical strip from $z=0$ to $z=1$. The second integration represents this strip sweeping across from $y=0$ to $y=1$ and is the integration over the slice that is swept out by the strip. Finally the integration with respect to x represents this slice sweeping from $x=0$ to $x=1$ and is the integration over the entire cube. The integral therefore becomes

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\, dzdydx$$

Solution (contd.)

(b) In the particular case where the function is $f(x, y, z) = y^2 + z^2$, the integral becomes

$$\int_0^1 \int_0^1 \int_0^1 (y^2 + z^2)\ dzdydx$$

The inner integral is

$$\int_0^1 (y^2 + z^2)\ dz = \left[y^2 z + \frac{1}{3}z^3\right]_{z=0}^1 = y^2 \times 1 + \frac{1}{3} \times 1 - y^2 \times 0 - \frac{1}{3} \times 0 = y^2 + \frac{1}{3}$$

This inner integral is now placed into the intermediate integral to give

$$\int_0^1 (y^2 + \frac{1}{3})\ dy = \left[\frac{1}{3}y^3 + \frac{1}{3}y\right]_{y=0}^1 = \frac{1}{3} \times 1^3 + \frac{1}{3} \times 1 - \frac{1}{3} \times 0^3 - \frac{1}{3} \times 0 = \frac{2}{3}$$

Finally, this intermediate integral can be placed into the outer integral to give

$$\int_0^1 \frac{2}{3}\ dx = \left[\frac{2}{3}x\right]_0^1 = \frac{2}{3} \times 1 - \frac{2}{3} \times 0 = \frac{2}{3}$$

Example 17

Evaluate $\displaystyle\int_0^1 \int_0^2 \int_0^3 8xyz\ dzdydx$. This represents an integral over the cuboid given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

Solution

The inner integral is given by integrating the function with respect to z while keeping x and y constant.

$$\int_0^3 8xyz\ dz = \left[\ 4xyz^2\ \right]_0^3 = 4xy \times 9 - 0 = 36xy$$

This result is now integrated with respect to y while keeping x constant:

$$\int_0^2 36xy\ dy = \left[\ 18xy\right]_0^2 = 18x \times 4 - 0 = 72x$$

Finally, this result is integrated with respect to x:

$$\int_0^1 72x\ dx = \left[\ 36x^2\right]_0^1 = 36 \times 1 - 0 = 36$$

Hence, $\displaystyle\int_0^1 \int_0^2 \int_0^3 8xyz\ dzdydx = 36$

More generally, the limits on the inner integral may be functions of the "intermediate" and "outer" variables and the limits on the intermediate integral may be functions of the "outer" variable.

Example 18

V is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 4$. (see Figure 24).

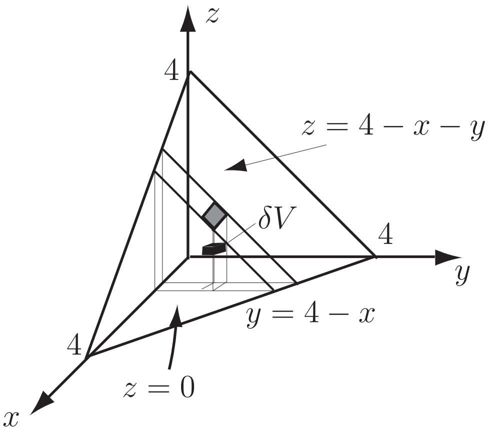


Figure 24

(a) Express $\int_V f(x, y, z)\, dV$ (where f is a function of x, y and z) as a triple integral.

(b) Hence find $\int_V x\, dV$.

Solution

The tetrahedron is divided into a series of slices parallel to the yz-plane and each slice is divided into a series of vertical strips. For each strip, the bottom is at $z = 0$ and the top is on the plane $x + y + z = 4$ i.e. $z = 4 - x - y$. So the integral up each strip is given by $\int_{z=0}^{4-x-y} f(x, y, z)\, dz$ and this (inner) integral will be a function of x and y.

This, in turn, is integrated over all strips which form the slice. For each value of x, one end of the slice will be at $y = 0$ and the other end at $y = 4 - x$. So the integral over the slice is $\int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdy$ and this (intermediate) integral will be a function of x.

Finally, integration is carried out over x. The limits on x are $x = 0$ and $x = 4$. Thus the triple integral is $\int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdydx$ and this (outer) integral will be a constant.

Hence $\int_V f(x, y, z)\, dV = \int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdydx.$

Solution (contd.)

In the case where $f(x,y,z) = x$, the integral becomes

$$\begin{aligned}
\int_V f(x,y,z)\,dV &= \int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} x\ dzdydx \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x}\Big[\ xz\ \Big]_{z=0}^{4-x-y}\ dydx \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x}\left[(4-x-y)x-0\right]\ dydx \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x}\left[4x-x^2-xy\right]\ dydx \\
&= \int_{x=0}^{4}\left[4xy-x^2y-\frac{1}{2}xy^2\right]_{y=0}^{4-x}\ dx \\
&= \int_{x=0}^{4}\left[4x(4-x)-x^2(4-x)-\frac{1}{2}x(4-x)^2-0\right]_{y=0}^{4-x}\ dx \\
&= \int_{x=0}^{4}\left[16x-4x^2-4x^2+x^3-8x+4x^2-\frac{1}{2}x^3\right]\ dx \\
&= \int_{x=0}^{4}\left[8x-4x^2+\frac{1}{2}x^3\right]\ dx \\
&= \left[4x^2-\frac{4}{3}x^3+\frac{1}{8}x^4\right]_0^4 = 4\times 4^2-\frac{4}{3}\times 4^3+\frac{1}{8}\times 4^4-0 \\
&= 64-\frac{256}{3}+32 \\
&= \frac{192-256+96}{3} \\
&= \frac{32}{3}
\end{aligned}$$

Key Point 7

Triple Integration

The procedure for carrying out a triple integral is very similar to that for a double integral except that the procedure requires three stages rather than two.

Example 19

Find the integral of x over the shape shown in Figure 25. It represents half (positive x) of a cylinder centered at $x = y = 0$ with radius 1 and vertical extent from $z = 0$ to $z = 1$.

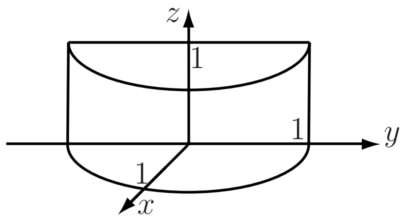


Figure 25

Solution

In terms of x, the shape goes from $x = 0$ to $x = 1$. For each value of x, y goes from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$. The variable z varies from $z = 0$ to $z = 1$. Hence the triple integral is

$$\begin{aligned} I &= \int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{1} x\ dzdydx \\ &= \int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\Big[xz\Big]_{z=0}^{1}\ dydx = \int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}[x-0]\ dydx = \int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\ dydx \\ &= \int_{x=0}^{1}\Big[xy\Big]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dx = \int_{x=0}^{1} 2x\sqrt{1-x^2}\ dx \end{aligned}$$

This outer integral can be evaluated by means of the substitution $U = 1 - x^2$ i.e. $dU = -2x\ dx$ and noting that $U = 1$ when $x = 0$ and $U = 0$ when $x = 1$ i.e.

$$I = \int_{x=0}^{1} 2x\sqrt{1-x^2}\ dx = -\int_{1}^{0} U^{1/2}dU = \int_{0}^{1} U^{1/2}dU = \left[\frac{2}{3}U^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

It is important to note that the three integrations can be carried out in whatever order is most convenient. The result does not depend on the order in which the integrals are carried out. However, when the order of the integrations is changed, it is necessary to consider carefully what the limits should be on each integration. Simply moving the limits from one integration to another will only work in the case of integration over a cuboid (i.e. where all limits are constants).

Key Point 8

Order of Integration for Triple Integrals

1. The three integrations can be carried out in whichever order is most convenient.
2. When changing the order of the integrations, it is important to reconsider the limits on each integration; a diagram can often help.

Example 20

For the triangular prism in Figure 26, with ends given by the planes $y = 0$ and $y = 2$ and remaining faces given by the planes $x = 0$, $z = 0$ and $x + 4z = 4$, find the integral of x over the prism, by

(a) Integrating first with respect to z, then y and finally x, and

(b) Changing the order of the integrations to x first, then y, then z.

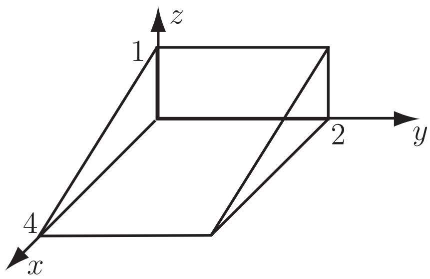


Figure 26

Solution

For every value of x and y, the vertical coordinate z varies from $z = 0$ to $z = 1 - x/4$. Hence the limits on z are $z = 0$ and $z = 1 - x/4$. For every value of x, the limits on y are $y = 0$ to $y = 2$. The limits on x are $x = 0$ and $x = 4$ (the limits on the figure). Hence the triple integral is $\displaystyle\int_0^4 \int_0^2 \int_0^{1-x/4} x\, dzdydx$ which can be evaluated as follows

$$
\begin{aligned}
I &= \int_0^4 \int_0^2 \int_0^{1-x/4} x\, dzdydx = \int_0^4 \int_0^2 \Big[xz \Big]_0^{1-x/4} dydx \\
&= \int_0^4 \int_0^2 \left[x\left(1 - \frac{x}{4}\right) - 0 \right] dydx = \int_0^4 \int_0^2 \left(x - \frac{1}{4}x^2 \right) dydx \\
&= \int_0^4 \left[\left(x - \frac{1}{4}x^2 \right) y \right]_0^2 dx = \int_0^4 \left[\left(x - \frac{1}{4}x^2 \right) \times 2 - \left(x - \frac{1}{4}x^2 \right) \times 0 \right] dx \\
&= \int_0^4 \left(2x - \frac{1}{2}x^2 \right) dx = \left[x^2 - \frac{1}{6}x^3 \right]_0^4 = 4^2 - \frac{1}{6} \times 4^3 - 0 = 16 - \frac{32}{3} = \frac{16}{3}
\end{aligned}
$$

Solution (contd.)

Now, if the order of the integrations is changed, it is necessary to re-derive the limits on the integrals. For every combination of y and z, x varies between $x = 0$ (left) and $x = 4 - 4z$ (right). Hence the limits on x are $x = 0$ and $x = 4 - 4z$. The limits on y are $y = 0$ and $y = 2$ (for all z). The limits of z are $z = 0$ (bottom) and $z = 1$ (top).

So the triple integral becomes $\displaystyle\int_0^1 \int_0^2 \int_0^{4-4z} x\, dxdydz$ which can be evaluated as follows

$$
\begin{aligned}
I &= \int_0^1 \int_0^2 \int_0^{4-4z} x\, dxdydz = \int_0^1 \int_0^2 \left[\frac{1}{2}x^2\right]_0^{4-4z} dydz \\
&= \int_0^1 \int_0^2 \left[\frac{1}{2}(4-4z)^2\right] dydz = \int_0^1 \int_0^2 (8 - 16z + 8z^2)\, dydz \\
&= \int_0^1 \left[\ (8 - 16z + 8z^2)\, y\right]_0^2 dz = 2\int_0^1 (8 - 16z + 8z^2)\, dz \\
&= 2\left[8z - 8z^2 + \frac{8}{3}z^3\right]_0^1 = 2\left(8 - 8 + \frac{8}{3} - 0\right) = \frac{16}{3}
\end{aligned}
$$

Key Point 9

Limits of Integration

While for different orders of integration the integral will always evaluate to the same value, the limits of integration will in general be different.

Task

Evaluate the triple integral: $\displaystyle\int_0^2 \int_0^3 \int_0^2 x^3 y^2 z\, dxdydz$

Your solution

Answer

The inner integral is

$$\int_0^2 x^3y^2z\,dx = \left[\frac{1}{4}x^4y^2z\right]_0^2 = \frac{1}{4}2^4y^2z - 0 = 4y^2z$$

This is put into the intermediate integral i.e.

$$\int_0^3 4y^2z\,dy = \left[\frac{4}{3}y^3z\right]_0^3 = \frac{4}{3}3^3z - 0 = 36z$$

Finally, this is put in the outer integral to give

$$I = \int_0^2 36z\,dz = \left[18z^2\right]_0^2 = 18 \times 2^2 - 0 = 72$$

Exercises

Evaluate the following triple integrals

1. $\displaystyle\int_0^2\int_0^x\int_0^{x+z}(x+y+z)\,dydzdx$

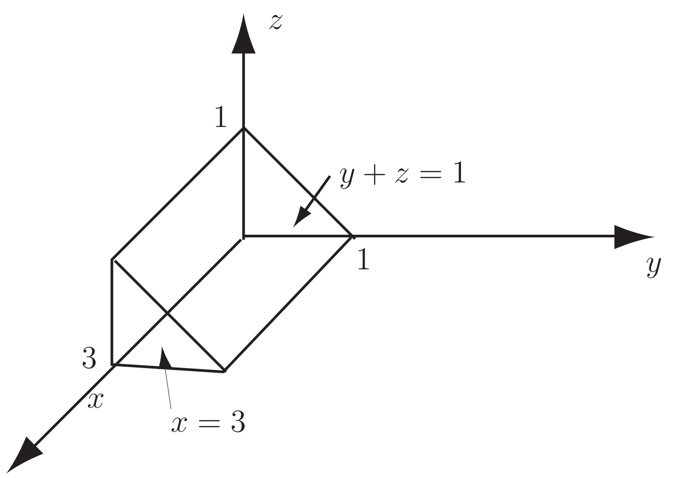
2. $\displaystyle\int_2^4\int_{-1}^3\int_{x/2-2}^{2-x/2}(x+y)\,dzdydx$

Answer

1. 14 2. $\dfrac{88}{3}$

Task

Find the volume of the solid prism shown in the diagram below. Check that when the order of integration is changed, the volume remains unaltered.



Your solution

Answer

The volume is given by the triple integral $\iiint dV$.

Putting z on the outer integral, y on the intermediate integral and x on the inner integral, the limits on z are $z = 0$ to $z = 1$. For each value of z, y varies from $y = 0$ (base) to $y = 1 - z$ on the sloping face. For each combination of y and z, x varies from $x = 0$ to $x = 3$. Thus, the volume is given by

$$\begin{aligned}
V &= \int\int\int dV = \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{3} dxdydz \\
&= \int_{z=0}^{1}\int_{y=0}^{1-z}\Big[x\Big]_0^3 dydz = \int_{z=0}^{1}\int_{y=0}^{1-z} 3\ dydz \\
&= \int_{z=0}^{1}\Big[3y\Big]_{y=0}^{1-z} dz = \int_{z=0}^{1}(3(1-z)-0)\ dz = \int_{z=0}^{1}(3-3z)\ dz \\
&= \left[3z - \frac{3}{2}z^2\right]_0^1 = 3 - \frac{3}{2} - (0-0) = \frac{3}{2} = 1.5
\end{aligned}$$

Answers continued

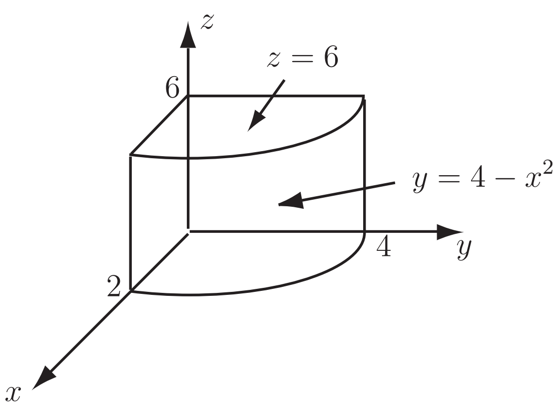
Now, the three integrations can be carried out in a different order. For example, with x on the outer integral, z on the intermediate integral and y on the inner integral, the limits on x are $x = 0$ to $x = 3$; for each value of x, z varies from $z = 0$ to $z = 1$ and for each combination of x and z, y varies from $y = 0$ to $y = 1 - z$. The volume is therefore given by

$$\begin{aligned}
V &= \int\int\int dV = \int_{x=0}^{3}\int_{z=0}^{1}\int_{y=0}^{1-z} dydzdx \\
&= \int_{x=0}^{3}\int_{z=0}^{1}\Big[y\Big]_{y=0}^{1-z} dzdx \\
&= \int_{x=0}^{3}\int_{z=0}^{1}[1-z]\ dzdx \\
&= \int_{x=0}^{3}\left[z-\frac{z^2}{2}\right]_{z=0}^{1} dx \\
&= \int_{x=0}^{3}\left[1-\frac{1^2}{2}-0\right] dx \\
&= \int_{x=0}^{3}\frac{1}{2}\ dx \\
&= \left[\frac{x}{2}\right]_{x=0}^{3} = \frac{3}{2}-0 = 1.5
\end{aligned}$$

There are in all six ways (3!) to order the three integrations; each order gives the same answer of 1.5.

Exercise

Find the volume of the solid shown in the diagram below. Check that when the order of integration is changed, the volume remains unaltered.



Answer

32

3. Higher order integrals

A function may be integrated over four or more variables. For example, the integral

$$\int_{w=0}^{1}\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1-x}(w+y)\,dzdydxdw$$

represents the function $w+y$ being integrated over the variables w, x, y and z. This is an example of a quadruple integral.

The methods of evaluating quadruple integrals are very similar to those for double and triple integrals. Start the integration from the inside and gradually work outwards. Quintuple (five variable) and higher-order integrals also exist and the techniques are similar.

Example 21

Evaluate the quadruple integral $\displaystyle\int_{w=0}^{1}\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1-x}(w+y)\,dzdydxdw$.

Solution

The first integral, with respect to z gives

$$\int_0^{1-x}(w+y)\,dz=\Big[\,(w+y)z\Big]_0^{1-x}=(w+y)(1-x)-0=(w+y)(1-x).$$

The second integral, with respect to y gives

$$\int_0^1(w+y)(1-x)\,dy=\left[\left(wy+\frac{1}{2}y^2\right)(1-x)\right]_0^1=\left(w+\frac{1}{2}\right)(1-x)-0=\left(w+\frac{1}{2}\right)(1-x).$$

The third integral, with respect to x gives

$$\int_0^1\left(w+\frac{1}{2}\right)(1-x)\,dx=\left[\left(w+\frac{1}{2}\right)\left(x-\frac{x^2}{2}\right)\right]_0^1=\left(w+\frac{1}{2}\right)\frac{1}{2}-0=\frac{1}{2}\left(w+\frac{1}{2}\right)=\frac{1}{2}w+\frac{1}{4}.$$

Finally, integrating with respect to w gives

$$\int_0^1\left(\frac{1}{2}w+\frac{1}{4}\right)dw=\left[\frac{1}{4}w^2+\frac{1}{4}w\right]_0^1=\frac{1}{4}+\frac{1}{4}-0=\frac{1}{2}$$

Exercise

Evaluate the quadruple integral $\displaystyle\int_0^1\int_{-1}^{1}\int_{-1}^{1}\int_0^{1-y^2}(x+y^2)\,dzdydxdw$.

Answer

$\dfrac{8}{15}$

4. Applications of triple and higher integrals

The integral $\int\int\int f(x,y,z)\ dzdydx$ (or $\int_V f(x,y,z)\ dV$) may represent many physical quantities depending on the function $f(x,y,z)$ and the limits used.

Volume

The integral $\int_V 1\ dV$ (i.e. the integral of the function $f(x,y,z)=1$) with appropriate limits gives the volume of the solid described by V. This is sometimes more convenient than finding the volume by means of a double integral.

Mass

The integral $\int\int\int \rho(x,y,z)\ dzdydx$ (or $\int_V \rho(x,y,z)\ dV$), with appropriate limits, gives the mass of the solid bounded by V.

Mass of water in a reservoir

The introduction to this Section concerned the mass of water in a reservoir. Imagine that the reservoir is rectangular in profile and that the width along the dam (i.e. measured in the x direction) is 100 m. Imagine also that the length of the reservoir (measured away from the dam i.e. in the y direction) is 400 m. The depth of the reservoir is given by $40 - y/10$ m i.e. the reservoir is 40 m deep along the dam and the depth reduces to zero at the end away from the dam.

The density of the water can be approximated by $\rho(z) = a - b \times z$ where $a = 998$ kg m^{-3} and $b = 0.05$ kg m^{-4}. I.e. at the surface ($z = 0$) the water has density 998 kg m^{-3} (corresponding to a temperature of $20°$C) while 40 m down i.e. $z = -40$, the water has a density of 1000 kg m^{-3} (corresponding to the lower temperature of $4°$C).

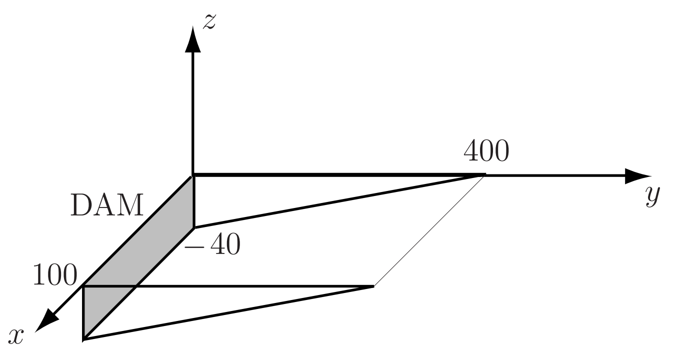


Figure 27

The mass of water in the reservoir is given by the integral of the function $\rho(z) = a - b \times z$. For each value of x and y, the limits on z will be from $y/10 - 40$ (bottom) to 0 (top). Limits on y will be 0 to 400 m while the limits of x will be 0 to 100 m. The mass of water is therefore given by the integral

$$M = \int_0^{100} \int_0^{400} \int_{y/10-40}^{0} (a - bz)\ dzdydx$$

which can be evaluated as follows

$$\begin{aligned}
M &= \int_0^{100}\int_0^{400}\int_{y/10-40}^{0} (a-bz)\, dzdydx \\
&= \int_0^{100}\int_0^{400}\left[az-\frac{b}{2}z^2\right]_{y/10-40}^{0} dydx \\
&= \int_0^{100}\int_0^{400}\left[0-a(y/10-40)+\frac{b}{2}(y/10-40)^2\right] dydx \\
&= \int_0^{100}\int_0^{400}\left[40a-\frac{a}{10}y+\frac{b}{200}y^2-4by+800b\right] dydx \\
&= \int_0^{100}\left[40ay-\frac{a}{20}y^2+\frac{b}{600}y^3-2by^2+800by\right]_0^{400} dx \\
&= \int_0^{100}\left[16000a-8000a+\frac{320000}{3}b-320000b+320000b\right] dx \\
&= \int_0^{100}\left[8000a+\frac{320000}{3}b\right] dx \\
&= 8\times 10^5 a+\frac{3.2}{3}\times 10^7 b = 7.984\times 10^8+\frac{0.16}{3}\times 10^7 = 7.989\times 10^8 \text{ kg}
\end{aligned}$$

So the mass of water in the reservoir is 7.989×10^8 kg.
Notes :

1. In practice, the profile of the reservoir would not be rectangular and the depth would not vary so smoothly.

2. The variation of the density of water with height is only a minor factor so it would only be taken into account when a very exact answer was required. Assuming that the water had a uniform density of $\rho = 998$ kg m^{-3} would give a total mass of 7.984×10^8 kg while assuming a uniform density of $\rho = 1000$ kg m^{-3} gives a total mass of 8×10^8 kg.

Centre of mass

The expressions for the centre of mass $(\overline{x},\ \overline{y},\ \overline{z})$ of a solid of density $\rho(x,y,z)$ are given below

$$\overline{x} = \frac{\displaystyle\int \rho(x,y,z)x\ dV}{\displaystyle\int \rho(x,y,z)\ dV} \qquad \overline{y} = \frac{\displaystyle\int \rho(x,y,z)y\ dV}{\displaystyle\int \rho(x,y,z)\ dV} \qquad \overline{z} = \frac{\displaystyle\int \rho(x,y,z)z\ dV}{\displaystyle\int \rho(x,y,z)\ dV}$$

In the (fairly common) case where the density ρ does not vary with position and is constant, these results simplify to

$$\overline{x} = \frac{\displaystyle\int x\ dV}{\displaystyle\int\ dV} \qquad \overline{y} = \frac{\displaystyle\int y\ dV}{\displaystyle\int\ dV} \qquad \overline{z} = \frac{\displaystyle\int z\ dV}{\displaystyle\int\ dV}$$

Example 22

A tetrahedron is enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 4$.
Find (a) the volume of this tetrahedron, (b) the position of the centre of mass.

Solution

(a) Note that this tetrahedron was considered in Example 18, see Figure 24. It was shown that in this case the volume integral $\displaystyle\int_V f(x, y, z)\, dV$ becomes $\displaystyle\int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} f(x, y, z)\, dzdydx$. The volume is given by

$$
\begin{aligned}
V &= \int_V dV = \int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} dzdydx \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x}\Big[\, z \,\Big]_{z=0}^{4-x-y} dydx \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x}(4 - x - y)\, dydx \\
&= \int_{x=0}^{4}\left[4y - xy - \frac{1}{2}y^2\right]_{y=0}^{4-x} dx \\
&= \int_{x=0}^{4}\left[8 - 4x + \frac{1}{2}x^2\right] dx \\
&= \left[8x - 2x^2 + \frac{1}{6}x^3\right]_0^4 = 32 - 32 + \frac{64}{6} = \frac{32}{3}
\end{aligned}
$$

Thus the volume of the tetrahedron is $\dfrac{32}{3} \approx 10.3$

(b) The x coordinate of the centre of mass i.e. $\overline{x}$ is given by $\overline{x} = \dfrac{\displaystyle\int x\, dV}{\displaystyle\int dV}$.

The denominator $\displaystyle\int dV$ is the formula for the volume i.e. $\dfrac{32}{3}$ while the numerator $\displaystyle\int x\, dV$ was calculated in an earlier Example to be $\dfrac{32}{3}$.

Thus $\overline{x} = \dfrac{\displaystyle\int x\, dV}{\displaystyle\int dV} = \dfrac{32/3}{32/3} = 1.$

By symmetry (or by evaluating relevant integrals), it can be shown that $\overline{y} = \overline{z} = 1$ i.e. the centre of mass is at $(1, 1, 1)$.

Moment of inertia

The moment of inertia I of a particle of mass M about an axis PQ is defined as

$$I = \text{Mass} \times \text{Distance}^2 \qquad \text{or} \qquad I = Md^2$$

where d is the perpendicular distance from the particle to the axis.
To find the moment of inertia of a larger object, it is necessary to carry out a volume integration over all such particles. The distance of a particle at (x, y, z) from the z-axis is given by $\sqrt{x^2 + y^2}$ so the moment of inertia of an object about the z-axis is given by

$$I_z = \int_V \rho(x, y, z)(x^2 + y^2)\; dz$$

Similarly, the moments of inertia about the x-axis and y-axis are given by

$$I_x = \int_V \rho(x, y, z)(y^2 + z^2)\; dx \qquad \text{and} \qquad I_y = \int_V \rho(x, y, z)(x^2 + z^2)\; dy$$

In the case where the density is constant over the object, so $\rho(x, y, z) = \rho$, these formulae reduce to

$$I_x = \rho \int_V (y^2 + z^2)\; dx \quad , \qquad I_y = \rho \int_V (x^2 + z^2)\; dy \qquad \text{and} \qquad I_z = \rho \int_V (x^2 + y^2)\; dz$$

When possible, the moment of inertia is expressed in terms of M, the mass of the object.

Example 23

Find the moment of inertia (about the x-axis) of the cube of side 1, mass M and density ρ shown in Example 16, page 43.

Solution

For the cube,

$$\text{Mass} = \text{Volume} \times \text{Density i.e.} \qquad M = 1^3 \times \rho = \rho$$

The moment of inertia (about the x-axis) is given by

$$I_x = \rho \int_V (y^2 + z^2)\; dx = \rho \int_0^1 \int_0^1 \int_0^1 (y^2 + z^2)\; dzdydx$$

This integral was shown to equal $\dfrac{2}{3}$ in Example 16. Thus

$$I_x = \frac{2}{3}\rho = \frac{2}{3}M$$

By applying symmetry, it can also be shown that the moments of inertia about the y- and z-axes are also equal to $\dfrac{2}{3}M$.

Engineering Example 2

Radioactive decay

Introduction

A cube of an impure radioactive ore is of side 10 cm. The number of radioactive decays taking place per cubic metre per second is given by $R = 10^{23}(0.1 - z)e^{-t/1000}$. The dependence on time represents a half-life of 693 seconds while the dependence on the vertical coordinate z represents some gravitational stratification. The value $z = 0$ represents the bottom of the cube and $z = 0.1$ represents the top of the cube. (Note that the dimensions are in metres so 10 cm becomes 0.1 m.)

What is the total number of decays taking place over the cube in the 100 seconds between $t = 0$ and $t = 100$?

Solution

The total number of decays is given by the quadruple integral

$$N = \int_{x=0}^{0.1}\int_{y=0}^{0.1}\int_{z=0}^{0.1}\int_{t=0}^{100} 10^{23}(0.1-z)e^{-t/1000}dtdzdydx$$

which may be evaluated as follows

$$\begin{aligned}
N &= \int_{x=0}^{0.1}\int_{y=0}^{0.1}\int_{z=0}^{0.1}\int_{t=0}^{100} 10^{23}(0.1-z)e^{-t/1000}dtdzdydx \\
&= \int_{x=0}^{0.1}\int_{y=0}^{0.1}\int_{z=0}^{0.1}\Big[-1000 \times 10^{23}(0.1-z)e^{-t/1000}\Big]_{t=0}^{100} dzdydx \\
&= \int_{x=0}^{0.1}\int_{y=0}^{0.1}\int_{z=0}^{0.1}\left[10^{26}(0.1-z)(1-e^{-0.1})\right] dzdydx \\
&= \int_{x=0}^{0.1}\int_{y=0}^{0.1}\int_{z=0}^{0.1}\left[9.5 \times 10^{24}(0.1-z)\right] dzdydx \\
&= 9.5 \times 10^{24}\int_{x=0}^{0.1}\int_{y=0}^{0.1}\Big[(0.1z - 0.5z^2)\Big]_{z=0}^{0.1} dydx \\
&= 9.5 \times 10^{24}\int_{x=0}^{0.1}\int_{y=0}^{0.1}[0.005]\ dydx \\
&= 0.005 \times 9.5 \times 10^{24}\int_{x=0}^{0.1}\int_{y=0}^{0.1} dydx \\
&= 0.005 \times 9.5 \times 10^{24} \times 0.1 \times 0.1 = 4.75 \times 10^{20}
\end{aligned}$$

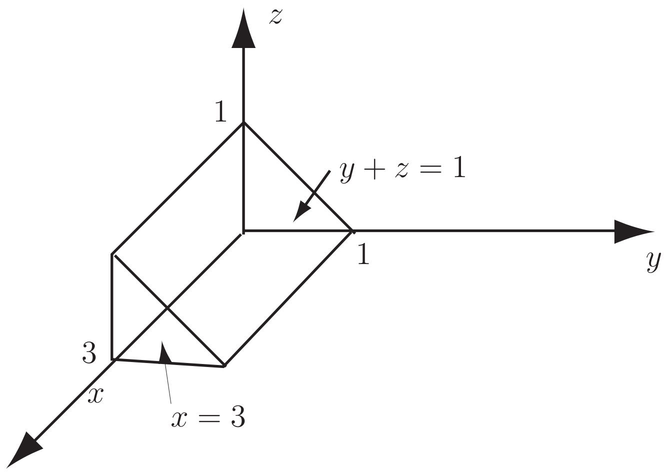
Thus the number of decays is approximately equal to 4.75×10^{20}

Task

For the solid prism shown below (the subject of the Task on page 50) find

(a) the coordinates of the centre of mass

(b) the moment of inertia about the x- , y- and z-axes.



Your solution

(a)

Answer

The x, y and z coordinates of the centre of mass of a solid of constant density are given on page 55 by

$$\overline{x} = \frac{\displaystyle\int x\, dV}{\displaystyle\int dV} \qquad \overline{y} = \frac{\displaystyle\int y\, dV}{\displaystyle\int dV} \qquad \overline{z} = \frac{\displaystyle\int z\, dV}{\displaystyle\int dV}$$

For the triangular prism, the task on page 50 showed that the denominator $\displaystyle\int dV$ has value 1.5.

The numerator of the expression for $\overline{x}$ is given by

$$\begin{aligned}
\int x\, dV &= \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{3} x\, dxdydz = \int_{z=0}^{1}\int_{y=0}^{1-z}\left[\frac{x^2}{2}\right]_0^3 dydz = \int_{z=0}^{1}\int_{y=0}^{1-z}\frac{9}{2}\, dydz \\
&= \int_{z=0}^{1}\left[\frac{9}{2}y\right]_{y=0}^{1-z} dz = \int_{z=0}^{1}\left(\frac{9}{2}(1-z)-0\right) dz = \int_{z=0}^{1}\left(\frac{9}{2}-\frac{9}{2}z\right) dz \\
&= \left[\frac{9}{2}z - \frac{9}{4}z^2\right]_0^1 = \frac{9}{2}-\frac{9}{4}-(0-0) = \frac{9}{4} = 2.25
\end{aligned}$$

So, $\overline{x} = \dfrac{2.25}{1.5} = 1.5$. By similar integration it can be shown that $\bar{y} = \dfrac{1}{3}$, $\bar{z} = \dfrac{1}{3}$.

Your solution

(b)

Answer

The moment of inertia about the $x-$axis, I_x is given by $I_x = \rho \int_V (y^2+z^2)\, dV$ which for the solid under consideration is given by

$$\begin{aligned} I_x = \rho \int_{x=0}^{3}\int_{y=0}^{1}\int_{z=0}^{1-y} (y^2+z^2)\, dzdydx &= \rho \int_{x=0}^{3}\int_{y=0}^{1} \left(y^2 - y^3 + \frac{(1-y)^3}{3} \right) dydx \\ &= \rho \int_{x=0}^{3} \frac{1}{6}\, dx = \frac{1}{2}\rho \end{aligned}$$

Now, the mass M of the solid is given by $M = \rho \times \text{Volume} = \frac{3}{2}\rho$ (where the volume had been calculated in a previous example) so

$$I_x = \frac{1}{2}\rho = \frac{1}{2}\rho \times \frac{M}{\frac{3}{2}\rho} = \frac{1}{3}M$$

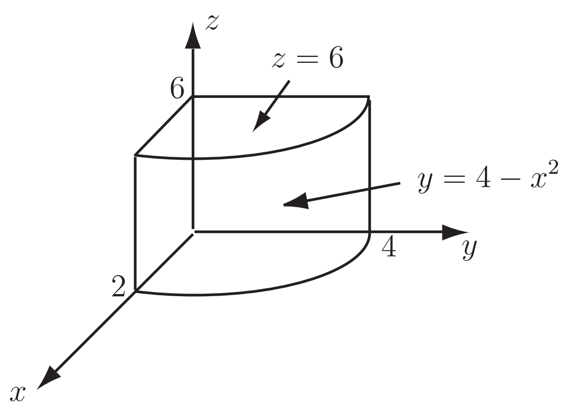
Similarly, the moment of inertia about the $y-$axis, I_y is given by $I_y = \rho \int_V (x^2+z^2)\, dV$ which for the solid under consideration is given by

$$\begin{aligned} I_x = \rho \int_{x=0}^{3}\int_{y=0}^{1}\int_{z=0}^{1-y} (x^2+z^2)\, dzdydx &= \rho \int_{x=0}^{3}\int_{y=0}^{1} \left(x^2(1-y) + \frac{(1-y)^3}{3} \right) dydx \\ &= \rho \int_{x=0}^{3} \left(\frac{1}{2}x^2 + \frac{1}{12} \right) dx = \frac{19}{4}\rho \end{aligned}$$

and so $I_y = \frac{19}{4}\rho = \frac{19}{4}\rho \times \frac{M}{\frac{3}{2}\rho} = \frac{19}{6}M$. Finally, by symmetry, $I_z = I_y = \frac{19}{6}M$.

Exercise

For the solid shown below (the subject of the Task on page 47) find the centre of mass and the moment of inertia about the x-, y- and z-axes.



Answer

$(\bar{x}, \bar{y}, \bar{z}) = (0.75, 1.6, 3) \qquad I_x = 15.66M \quad I_y = 12.8M \quad I_z = 4.46M$

Task

A cube of side 2 is made of laminated material so that, with the origin at one corner, the density of the material is kx.

(a) First find the mass M of the cube:

Your solution

Answer

The integrations over the cube are of the form $\displaystyle\int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} dV$.

The mass M is given by

$$\begin{aligned} M &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} \rho \, dzdydx \\ &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} kx \, dzdydx \\ &= \int_{x=0}^{2}\int_{y=0}^{2} 2kx \, dydx = \int_{x=0}^{2} 4kx \, dx = \left[2kx^2\right]_0^2 = 8k \end{aligned}$$

(b) Now find the position of the centre of mass of the cube:

Your solution

Answer

The x-coordinate of the centre of mass will be given by $\dfrac{\displaystyle\int \rho x\ dV}{M}$ where the numerator is given by

$$\begin{aligned}\int \rho x dV &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} \rho x\ dzdydx = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} kx^2\ dzdydx \\ &= \int_{x=0}^{2}\int_{y=0}^{2} 2kx^2\ dydx = \int_{x=0}^{2} 4kx^2\ dx = \left[\frac{4}{3}kx^3\right]_0^2 = \frac{32}{3}k\end{aligned}$$

So $\overline{x} = \dfrac{\frac{32}{3}k}{8k} = \dfrac{4}{3}$.

The y-coordinate of the centre of mass is given by $\dfrac{\displaystyle\int \rho y\ dV}{M}$ where the numerator is given by

$$\begin{aligned}\int \rho y dV &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} \rho y\ dzdydx = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} kxy\ dzdydx \\ &= \int_{x=0}^{2}\int_{y=0}^{2} 2kxy\ dydx = \int_{x=0}^{2} 4kx\ dx = \left[\ 2kx^2\right]_0^2 = 8k\end{aligned}$$

So $\overline{y} = \dfrac{8k}{8k} = 1$.
By symmetry (the density depends only on x), $\overline{z} = \overline{y} = 1$.

The coordinates of the centre of mass are $(\dfrac{4}{3}, 1, 1)$.

(c) Finally find the moments of inertia about the x-, y- and z-axes:

Your solution

Answer

The moment of inertia about the x-axis is given by $I_x = \int_V \rho\,(y^2+z^2)\ dV$ (page 58). In this case,

$$\begin{aligned}
I_x &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} kx(y^2+z^2)\ dzdydx \\
&= \int_{x=0}^{2}\int_{y=0}^{2}\left[kx(y^2z+\frac{1}{3}z^3)\right]_{z=0}^{2}\ dydx = \int_{x=0}^{2}\int_{y=0}^{2} kx(2y^2+\frac{8}{3})\ dydx \\
&= \int_{x=0}^{2}\left[kx(\frac{2}{3}y^3+\frac{8}{3}y)\right]_{y=0}^{2}\ dx = \int_{x=0}^{2} kx(\frac{32}{3})\ dx \\
&= \left[\frac{16}{3}kx^2\right]_0^2 = \frac{64}{3}k = \frac{8}{3}M
\end{aligned}$$

where the last step involves substituting that the mass $M = 8k$.

Similarly, the moment of inertia about the y-axis is given by $I_y = \int_V \rho\,(x^2+z^2)\ dV$ i.e.

$$\begin{aligned}
I_y &= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} kx(x^2+z^2)\ dzdydx \\
&= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} k(x^3+xz^2)\ dzdydx = \int_{x=0}^{2}\int_{y=0}^{2}\left[k(x^3z+\frac{1}{3}xz^3)\right]_{z=0}^{2}\ dydx \\
&= \int_{x=0}^{2}\int_{y=0}^{2}\left(k(2x^3+\frac{8}{3}x)\right)\ dydx = \int_{x=0}^{2}\left(k(4x^3+\frac{16}{3}x)\right)\ dx \\
&= \left[k(x^4+\frac{8}{3}x^2)\right]_{x=0}^{2} = k(16+\frac{32}{3}) = \frac{80}{3}k = \frac{10}{3}M
\end{aligned}$$

By symmetry, $I_z = I_y = \frac{10}{3}M$.

Changing Coordinates 27.4

Introduction

We have seen how changing the variable of integration of a single integral or changing the coordinate system for multiple integrals can make integrals easier to evaluate. In this Section we introduce the Jacobian. The Jacobian gives a general method for transforming the coordinates of any multiple integral.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of several variables
- be able to evaluate the determinant of a matrix

Learning Outcomes

On completion you should be able to ...

- decide which coordinate transformation simplifies an integral
- determine the Jacobian for a coordinate transformation
- evaluate multiple integrals using a transformation of coordinates

1. Changing variables in multiple integrals

When the method of substitution is used to solve an integral of the form $\displaystyle\int_a^b f(x)\,dx$ three parts of the integral are changed, the limits, the function and the infinitesimal dx. So if the substitution is of the form $x = x(u)$ the u limits, c and d, are found by solving $a = x(c)$ and $b = x(d)$ and the function is expressed in terms of u as $f(x(u))$.

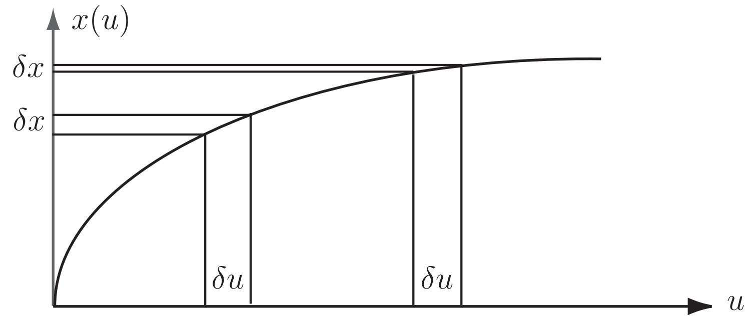


Figure 28

Figure 28 shows why the dx needs to be changed. While the δu is the same length for all u, the δx change as u changes. The rate at which they change is precisely $\dfrac{d}{du}x(u)$. This gives the relation

$$\delta x = \frac{dx}{du}\delta u$$

Hence the transformed integral can be written as

$$\int_a^b f(x)\,dx = \int_c^d f\left(x\left(u\right)\right)\frac{dx}{du}du$$

Here the $\dfrac{dx}{du}$ is playing the part of the Jacobian that we will define.

Another change of coordinates that you have seen is the transformations from cartesian coordinates (x, y) to polar coordinates (r, θ).
Recall that a double integral in polar coordinates is expressed as

$$\int\!\!\int f(x,y)\,dxdy = \int\!\!\int g(r,\theta)\,rdrd\theta$$

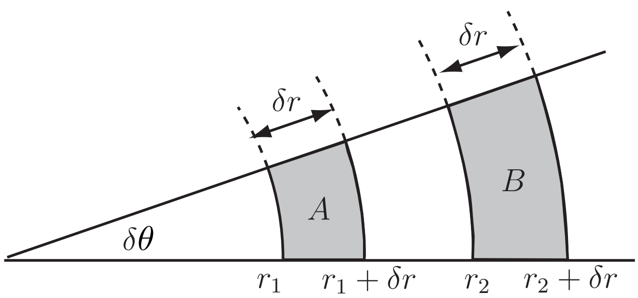


Figure 29

We can see from Figure 29 that the area elements change in size as r increases. The circumference of a circle of radius r is $2\pi r$, so the length of an arc spanned by an angle θ is $2\pi r\dfrac{\theta}{2\pi} = r\theta$. Hence the area elements in polar coordinates are approximated by rectangles of width δr and length $r\delta\theta$. Thus under the transformation from cartesian to polar coordinates we have the relation

$$\delta x \delta y \to r\delta r \delta\theta$$

that is, $r\delta r\delta\theta$ plays the same role as $\delta x\delta y$. This is why the r term appears in the integrand. Here r is playing the part of the Jacobian.

2. The Jacobian

Given an integral of the form $\displaystyle\int\int_A f(x,y)\,dxdy$

Assume we have a change of variables of the form $x = x(u,v)$ and $y = y(u,v)$ then the Jacobian of the transformation is defined as

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Key Point 10

Jacobian in Two Variables

For given transformations $x = x(u,v)$ and $y = y(u,v)$ the Jacobian is

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Notice the pattern occurring in the x, y, u and v. Across a row of the determinant the numerators are the same and down a column the denominators are the same.

Notation

Different textbooks use different notation for the Jacobian. The following are equivalent.

$$J(u,v) = J(x,y;u,v) = J\left(\frac{x,y}{u,v}\right) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

The Jacobian correctly describes how area elements change under such a transformation. The required relationship is

$$dxdy \to |J(u,v)|\ dudv$$

that is, $|J(u,v)|\ dudv$ plays the role of $dxdy$.

Key Point 11

Jacobian for Transforming Areas

When transforming area elements employing the Jacobian it is the **modulus** of the Jacobian that must be used.

Example 24

Find the area of the circle of radius R.

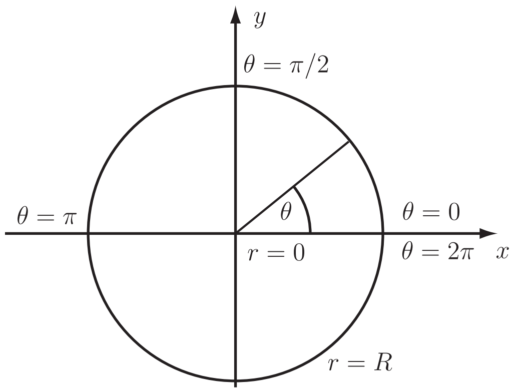


Figure 30

Solution

Let A be the region bounded by a circle of radius R centred at the origin. Then the area of this region is $\int_A dA$. We will calculate this area by changing to polar coordinates, so consider the usual transformation $x = r\cos\theta, y = r\sin\theta$ from cartesian to polar coordinates. First we require all the partial derivatives

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Thus

$$\begin{aligned} J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} &= \cos\theta \times r\cos\theta - (-r\sin\theta)\times\sin\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r \end{aligned}$$

Solution (contd.)

This confirms the previous result for polar coordinates, $dxdy \to rdrd\theta$. The limits on r are $r = 0$ (centre) to $r = R$ (edge). The limits on θ are $\theta = 0$ to $\theta = 2\pi$, i.e. starting to the right and going once round anticlockwise. The required area is

$$\int_A dA = \int_0^{2\pi}\int_0^R |J(r,\theta)|\, drd\theta = \int_0^{2\pi}\int_0^R r\, drd\theta = 2\pi\frac{R^2}{2} = \pi R^2$$

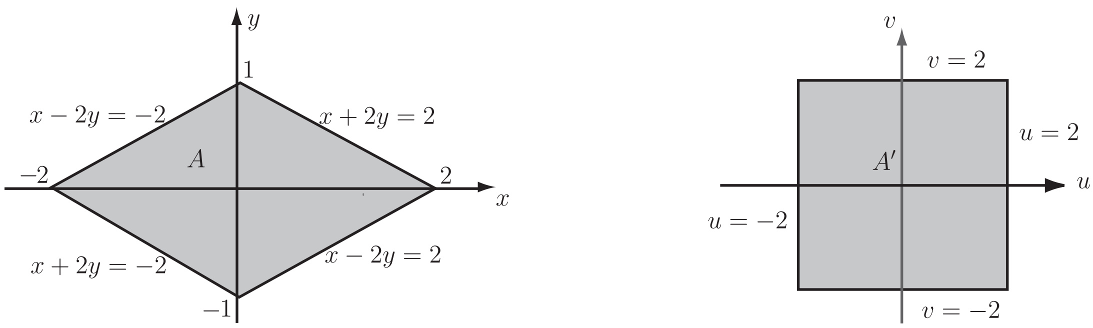
Note that here $r > 0$ so $|J(r,\theta)| = J(r,\theta) = r$.

Example 25

The diamond shaped region A in Figure 31(a) is bounded by the lines $x + 2y = 2$, $x - 2y = 2$, $x + 2y = -2$ and $x - 2y = -2$. We wish to evaluate the integral

$$I = \iint_A (3x + 6y)^2\, dA$$

over this region. Since the region A is neither vertically nor horizontally simple, evaluating I without changing coordinates would require separating the region into two simple triangular regions. So we use a change of coordinates to transform A to a square region in Figure 31(b) and evaluate I.



(a) (b)

Figure 31

Solution

By considering the equations of the boundary lines of region A it is easy to see that the change of coordinates

$$du = x + 2y \qquad (1) \qquad\qquad v = x - 2y \qquad (2)$$

will transform the boundary lines to $u = 2$, $u = -2$, $v = 2$ and $v = -2$. These values of u and v are the new limits of integration. The region A will be transformed to the square region A' shown above.

We require the inverse transformations so that we can substitute for x and y in terms of u and v. By adding (1) and (2) we obtain $u + v = 2x$ and by subtracting (1) and (2) we obtain $u - v = 4y$, thus the required change of coordinates is

$$x = \frac{1}{2}(u + v) \qquad y = \frac{1}{4}(u - v)$$

Substituting for x and y in the integrand $(3x + 6y)^2$ of I gives

$$\left(\tfrac{3}{2}(u + v) + \tfrac{6}{4}(u - v)\right)^2 = 9u^2$$

We have the new limits of integration and the new form of the integrand, we now require the Jacobian. The required partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial x}{\partial v} = \frac{1}{2} \qquad \frac{\partial y}{\partial u} = \frac{1}{4} \qquad \frac{\partial y}{\partial v} = -\frac{1}{4}$$

Then the Jacobian is

$$J(u, v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

Then $dA' = |J(u, v)|dA = \dfrac{1}{4}dA$. Using the new limits, integrand and the Jacobian, the integral can be written

$$I = \int_{-2}^{2} \int_{-2}^{2} \frac{9}{4}u^2 \, dudv.$$

You should evaluate this integral and check that $I = 48$.

This Task concerns using a transformation to evaluate $\iint (x^2 + y^2)\, dxdy$.

(a) Given the transformations $u = x + y$, $v = x - y$ express x and y in terms of u and v to find the inverse transformations:

Your solution

Answer

$$u = x + y \qquad (1)$$

$$v = x - y \qquad (2)$$

Add equations (1) and (2) $\quad u + v = 2x$

Subtract equation (2) from equation (1) $\quad u - v = 2y$

So $\quad x = \frac{1}{2}(u + v) \qquad\qquad y = \frac{1}{2}(u - v)$

(b) Find the Jacobian $J(u, v)$ for the transformation in part (a):

Your solution

Answer

Evaluating the partial derivatives, $\dfrac{\partial x}{\partial u} = \dfrac{1}{2}$, $\dfrac{\partial x}{\partial v} = \dfrac{1}{2}$, $\dfrac{\partial y}{\partial u} = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial v} = -\dfrac{1}{2}$ so the Jacobian

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

(c) Express the integral $I = \int\int \left(x^2 + y^2\right) \ dxdy$ in terms of u and v, using the transformations introduced in (a) and the Jacobian found in (b):

Your solution

Answer

On letting $x = \dfrac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$ and $dxdy = |J| \ dudv = \frac{1}{2} \ dudv$, the integral $\int\int \left(x^2 + y^2\right) \ dxdy$ becomes

$$\begin{aligned} I &= \int\int \left(\frac{1}{4}(u+v)^2 + \frac{1}{4}(u-v)^2\right) \times \frac{1}{2} \, dudv \\ &= \int\int \frac{1}{2}\left(u^2+v^2\right) \times \frac{1}{2} \, dudv \\ &= \int\int \frac{1}{4}\left(u^2+v^2\right) \ dudv \end{aligned}$$

(d) Find the limits on u and v for the rectangle with vertices $(x,y) = (0,0), \ (2,2), \ (-1,5), \ (-3,3)$:

Your solution

Answer

For $(0,0)$, $u = 0$ and $v = 0$
For $(2,2)$, $u = 4$ and $v = 0$
For $(-1,5)$, $u = 4$ and $v = -6$
For $(-3,3)$, $u = 0$ and $v = -6$
Thus, the limits on u are $u = 0$ to $u = 4$ while the limits on v are $v = -6$ to $v = 0$.

(e) Finally evaluate I:

Your solution

Answer
The integral is

$$\begin{aligned}
I &= \int_{v=-6}^{0}\int_{u=0}^{4} \frac{1}{4}\left(u^2+v^2\right)\, dudv \\
&= \frac{1}{4}\int_{v=-6}^{0}\left[\frac{1}{3}u^3+uv^2\right]_{u=0}^{4} dudv = \int_{v=-6}^{0}\left[\frac{16}{3}+v^2\right] dv \\
&= \left[\frac{16}{3}v+\frac{1}{3}v^3\right]_{-6}^{0} = 0-\left[\frac{16}{3}\times(-6)+\frac{1}{3}\times(-216)\right] = 104
\end{aligned}$$

3. The Jacobian in 3 dimensions

When changing the coordinate system of a triple integral

$$I = \iiint_V f(x,y,z)\, dV$$

we need to extend the above definition of the Jacobian to 3 dimensions.

Key Point 12

Jacobian in Three Variables

For given transformations $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ the Jacobian is

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

The same pattern persists as in the 2-dimensional case (see Key Point 10). Across a row of the determinant the numerators are the same and down a column the denominators are the same.

The volume element $dV = dxdydz$ becomes $dV = |J(u,v,w)|\ dudvdw$. As before the limits and integrand must also be transformed.

Example 26

Use spherical coordinates to find the volume of a sphere of radius R.

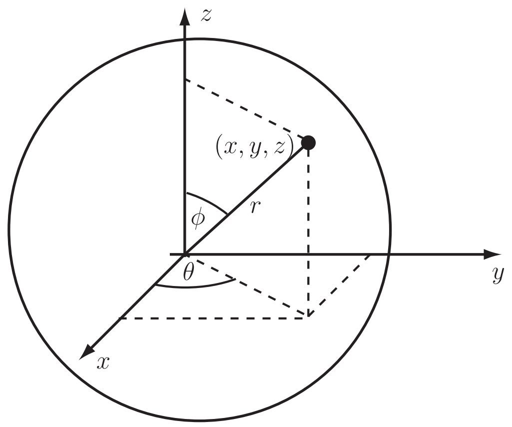


Figure 32

Solution

The change of coordinates from Cartesian to spherical polar coordinates is given by the transformation equations

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi$$

We now need the nine partial derivatives

$$\begin{array}{lll} \dfrac{\partial x}{\partial r} = \cos\theta\sin\phi & \dfrac{\partial x}{\partial \theta} = -r\sin\theta\sin\phi & \dfrac{\partial x}{\partial \phi} = r\cos\theta\cos\phi \\ \dfrac{\partial y}{\partial r} = \sin\theta\sin\phi & \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi & \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \\ \dfrac{\partial z}{\partial r} = \cos\phi & \dfrac{\partial z}{\partial \theta} = 0 & \dfrac{\partial z}{\partial \phi} = r\sin\phi \end{array}$$

Hence we have

$$J(r,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$J(r,\theta,\phi) = \cos\phi \begin{vmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + 0 - r\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi \end{vmatrix}$$

Check that this gives $J(r,\theta,\phi) = -r^2\sin\phi$. Notice that $J(r,\theta,\phi) \leq 0$ for $0 \leq \phi \leq \pi$, so $|J(r,\theta,\phi)| = r^2\sin\phi$. The limits are found as follows. The variable ϕ is related to 'latitude' with $\phi = 0$ representing the 'North Pole' with $\phi = \pi/2$ representing the equator and $\phi = \pi$ representing the 'South Pole'.

Solution (contd.)

The variable θ is related to 'longitude' with values of 0 to 2π covering every point for each value of ϕ. Thus limits on ϕ are 0 to π and limits on θ are 0 to 2π. The limits on r are $r = 0$ (centre) to $r = R$ (surface).

To find the volume of the sphere we then integrate the volume element $dV = r^2 \sin\phi \; dr d\theta d\phi$ between these limits.

$$\begin{aligned}\text{Volume} = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\phi \; dr d\theta d\phi \; &= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}R^3 \sin\phi \; d\theta d\phi \\ &= \int_0^{\pi} \frac{2\pi}{3}R^3 \sin\phi \; d\phi = \frac{4}{3}\pi R^3\end{aligned}$$

Example 27

Find the volume integral of the function $f(x, y, z) = x - y$ over the parallelepiped with the vertices of the base at

$$(x, y, z) = (0, 0, 0), (2, 0, 0), (3, 1, 0) \text{ and } (1, 1, 0)$$

and the vertices of the upper face at

$$(x, y, z) = (0, 1, 2),\ (2, 1, 2),\ (3, 2, 2) \text{ and } (1, 2, 2).$$

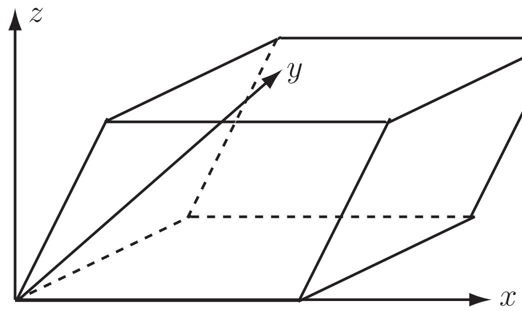


Figure 33

Solution

This will be a difficult integral to derive limits for in terms of x, y and z. However, it can be noted that the base is described by $z = 0$ while the upper face is described by $z = 2$. Similarly, the front face is described by $2y - z = 0$ with the back face being described by $2y - z = 2$. Finally the left face satisfies $2x - 2y + z = 0$ while the right face satisfies $2x - 2y + z = 4$.

The above suggests a change of variable with the new variables satisfying $u = 2x-2y+z$, $v = 2y-z$ and $w = z$ and the limits on u being 0 to 4, the limits on v being 0 to 2 and the limits on w being 0 to 2.

Inverting the relationship between u, v, w and x, y and z, gives

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(v+w) \qquad z = w$$

The Jacobian is given by

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} & \dfrac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{4}$$

Note that the function $f(x,y,z) = x - y$ equals $\frac{1}{2}(u+v) - \frac{1}{2}(v+w) = \frac{1}{2}(u-w)$. Thus the integral is

$$\begin{aligned}
\int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{2}(u-w)\frac{1}{4}\,dudvdw &= \int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{8}(u-w)\,dudvdw \\
&= \int_{w=0}^{2}\int_{v=0}^{2} \left[\frac{1}{16}u^2 - \frac{1}{8}uw\right]_0^4 dvdw \\
&= \int_{w=0}^{2}\int_{v=0}^{2} \left(1 - \frac{1}{2}w\right) dvdw \\
&= \int_{w=0}^{2} \left[v - \frac{vw}{2}\right]_0^2 dw \\
&= \int_{w=0}^{2} (2-w)\,dw \\
&= \left[2w - \frac{1}{2}w^2\right]_0^2 \\
&= 4 - \frac{4}{2} - 0 \\
&= 2
\end{aligned}$$

Task

Find the Jacobian for the following transformation:

$$x = 2u + 3v - w,\ y = v - 5w,\ z = u + 4w$$

Your solution

Answer

Evaluating the partial derivatives,

$$\begin{aligned}
&\frac{\partial x}{\partial u} = 2, \qquad &&\frac{\partial x}{\partial v} = 3, \qquad &&\frac{\partial x}{\partial w} = -1,\\
&\frac{\partial y}{\partial u} = 0, \qquad &&\frac{\partial y}{\partial v} = 1, \qquad &&\frac{\partial y}{\partial w} = -5,\\
&\frac{\partial z}{\partial u} = 1, \qquad &&\frac{\partial z}{\partial v} = 0, \qquad &&\frac{\partial z}{\partial w} = 4
\end{aligned}$$

so the Jacobian is

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & -5 \\ 1 & 0 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -5 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 1 & -5 \end{vmatrix} = 2 \times 4 + 1 \times (-14) = -6$$

where expansion of the determinant has taken place down the first column.

Engineering Example 3

Volume of liquid in an ellipsoidal tank

Introduction

An ellipsoidal tank (elliptical when viewed from along x-, y- or z-axes) has a volume of liquid poured into it. It is useful to know in advance how deep the liquid will be. In order to make this calculation, it is necessary to perform a multiple integration and calculate a Jacobian.

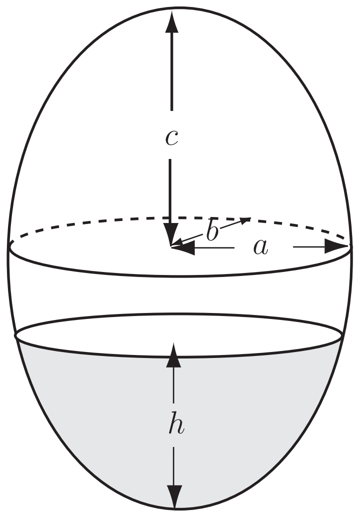


Figure 34

Problem in words

The metal tank is in the form of an ellipsoid, with semi-axes a, b and c. A volume V of liquid is poured into the tank ($V < \frac{4}{3}\pi abc$, the volume of the ellipsoid) and the problem is to calculate the depth, h, of the liquid.

Mathematical statement of problem

The shaded area is expressed as the triple integral

$$V = \int_{z=0}^{h} \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} dxdydz$$

where limits of integration

$$x_1 = -a\sqrt{1 - \frac{y^2}{b^2} - \frac{(z-c)^2}{c^2}} \quad \text{and} \quad x_2 = +a\sqrt{1 - \frac{y^2}{b^2} - \frac{(z-c)^2}{c^2}}$$

which come from rearranging the equation of the ellipsoid $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1\right)$ and limits

$$y_1 = -\frac{b}{c}\sqrt{c^2 - (z-c)^2} \quad \text{and} \quad y_2 = +\frac{b}{c}\sqrt{c^2 - (z-c)^2}$$

from the equation of an ellipse in the y-z plane $\left(\frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1\right)$.

Mathematical analysis

To calculate V, use the substitutions

$$\begin{aligned} x &= a\tau\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \\ y &= b\tau\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \\ z &= z \end{aligned}$$

now expressing the triple integral as

$$V = \int_{z=0}^{h}\int_{\phi=\phi_1}^{\phi_2}\int_{\tau=\tau_1}^{\tau_2} J\ d\tau d\phi dz$$

where J is the Jacobian of the transformation calculated from

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \tau} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \tau} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \tau} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix}$$

and reduces to

$$\begin{aligned} J &= \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \tau} \qquad \text{since } \frac{\partial z}{\partial \tau} = \frac{\partial z}{\partial \phi} = 0 \\ &= \left\{a\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \ \ b\tau\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\right\} \\ &\quad - \left\{-a\tau\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \ \ b\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\right\} \\ &= ab\tau\left(\cos^2\phi+\sin^2\phi\right)\left(1-\frac{(z-c)^2}{c^2}\right) \\ &= ab\tau\left(1-\frac{(z-c)^2}{c^2}\right) \end{aligned}$$

To determine limits of integration for ϕ, note that the substitutions above are similar to a cylindrical polar co-ordinate system, and so ϕ goes from 0 to 2π. For τ, setting $\tau = 0 \Rightarrow x = 0$ and $y = 0$, i.e. the z-axis.

Setting $\tau = 1$ gives

$$\frac{x^2}{a^2} = \cos^2\phi\left(1-\frac{(z-c)^2}{c^2}\right) \qquad (1)$$

and

$$\frac{y^2}{b^2} = \sin^2\phi\left(1-\frac{(z-c)^2}{c^2}\right) \qquad (2)$$

Summing both sides of Equations (1) and (2) gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (\cos^2\phi + \sin^2\phi)\left(1 - \frac{(z-c)^2}{c^2}\right)$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1$$

which is the equation of the ellipsoid, i.e. the outer edge of the volume. Therefore the range of τ should be 0 to 1. Now

$$\begin{aligned} V &= ab\int_{z=0}^{h}\left(1 - \frac{(z-c)^2}{c^2}\right)\int_{\phi=0}^{2\pi}\int_{\tau=0}^{1}\tau\ d\tau d\phi dz \\ &= \frac{ab}{c^2}\int_{z=0}^{h}(2zc - z^2)\int_{\phi=0}^{2\pi}\left[\frac{\tau^2}{2}\right]_{\tau=0}^{1} d\phi dz \\ &= \frac{ab}{2c^2}\int_{z=0}^{h}(2zc - z^2)\Big[\phi\Big]_{\phi=0}^{2\pi} dz \\ &= \frac{\pi ab}{c^2}\left[cz^2 - \frac{z^3}{3}\right]_{z=0}^{h} \\ &= \frac{\pi ab}{c^2}\left(ch^2 - \frac{h^3}{3}\right) \end{aligned}$$

Interpretation

Suppose the tank has actual dimensions of $a = 2$ m, $b = 0.5$ m and $c = 3$ m and a volume of 7 m^3 is to be poured into it. (The total volume of the tank is $4\pi\ m^3 \approx 12.57$ m^3). Then, from above

$$V = \frac{\pi ab}{c^2}\left(ch^2 - \frac{h^3}{3}\right)$$

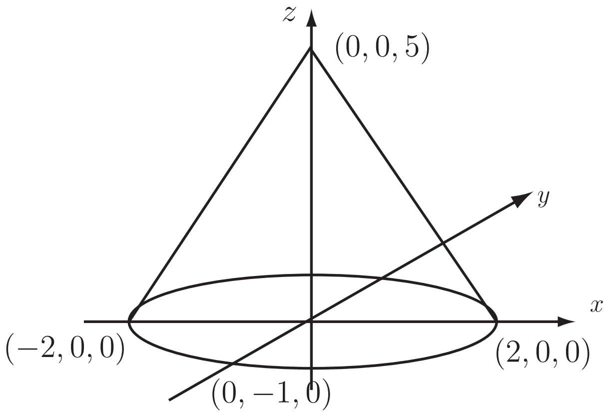
which becomes

$$7 = \frac{\pi}{9}\left(3h^2 - \frac{h^3}{3}\right)$$

with solution $h = 3.23$ m (2 d.p.), compared to the maximum height of the ellipsoid of 6 m.

Exercises

1. The function $f = x^2 + y^2$ is to be integrated over an elliptical cone with base being the ellipse, $x^2/4 + y^2 = 1$, $z = 0$ and apex (point) at $(0, 0, 5)$. The integral can be made simpler by means of the change of variables $x = 2(1 - \dfrac{w}{5})\tau\cos\theta$, $y = (1 - \dfrac{w}{5})\tau\sin\theta$, $z = w$.



   (a) Find the limits on the variables τ, θ and w.

   (b) Find the Jacobian $J(\tau, \theta, w)$ for this transformation.

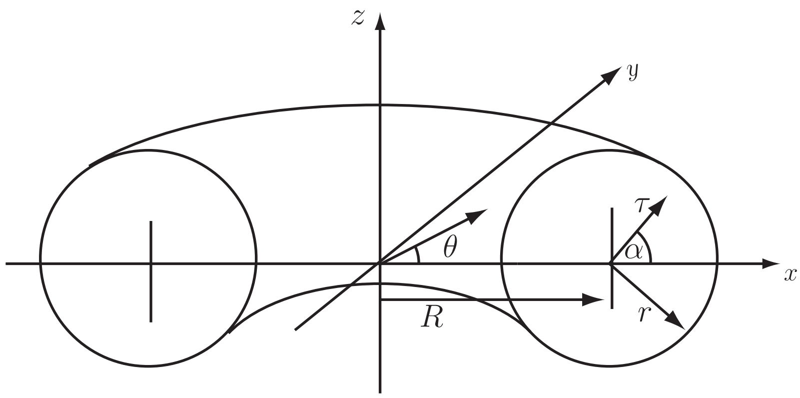
   (c) Express the integral $\displaystyle\iiint (x^2 + y^2)\ dxdydz$ in terms of τ, θ and w.

   (d) Evaluate this integral. [Hint:- it may be worth noting that $\cos^2\theta \equiv \frac{1}{2}(1 + \cos 2\theta)$].

   Note: This integral has relevance in topics such as moments of inertia.

2. Using cylindrical polar coordinates, integrate the function $f = z\sqrt{x^2 + y^2}$ over the volume between the surfaces $z = 0$ and $z = 1 + x^2 + y^2$ for $0 \le x^2 + y^2 \le 1$.

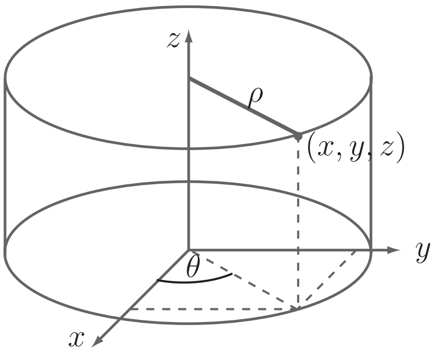
3. A torus (doughnut) has major radius R and minor radius r. Using the transformation $x = (R + \tau\cos\alpha)\cos\theta$, $y = (R + \tau\cos\alpha)\sin\theta$, $z = \tau\sin\alpha$, find the volume of the torus. [Hints:- limits on α and θ are 0 to 2π, limits on τ are 0 to r. Show that Jacobian is $\tau(R + \tau\cos\alpha)$].



4. Find the Jacobian for the following transformations.

   (a) $x = u^2 + vw$, $y = 2v + u^2 w$, $z = uvw$

   (b) Cylindrical polar coordinates. $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$



Answers

1. (a) τ : 0 to 1, θ : 0 to 2π, w : 0 to 5

   (b) $2(1 - \frac{w}{5})^2\tau$

   (c) $2\int_{\tau=0}^{1}\int_{\theta=0}^{2\pi}\int_{w=0}^{5}(1 - \frac{w}{5})^4\tau^3(4\cos^2\theta + \sin^2\theta)\, dwd\theta d\tau$

   (d) $\frac{5}{2}\pi$

2. $\frac{92}{105}\pi$

3. $2\pi^2 Rr^2$

4. (a) $4u^2v - 2u^4w + u^2vw^2 - 2v^2w$, (b) ρ